U. S. ARMY TEST AND EVALUATION COMMAND
BACKGROUND DOCUMENT

DYNAMIC STRUCTURAL DATA ANALYSIS

1. INTRODUCTION

This document describes methods and procedures that deal with the reduction, presentation, and analysis of environmental data which have specific application to structural evaluations and fall under the following categories:

a. Vibration
b. Shock
c. Acoustics
d. Strain

Engineers and other personnel involved in structural data analysis are aware that no one method of reduction or presentation is always the best one. Usually, the correct method depends on the type of data and the use to be made of the results.

The data analyst, generally, has the necessary equipment at his disposal to obtain the data he desires. However, the equipment limitations and characteristics must be weighed carefully against conflicting (usually mathematical) requirements. For this reason, the mathematics involved in the analysis are integrated into the operational discussion as much as possible. In any type of data analysis, knowledge of the mathematical formulae which govern the processes involved will produce a clearer insight into the analysis (or analyses) being conducted.

Basically, this document deals only with the essential background material and methods used in the analysis of a prerecorded signal. It is assumed that the results of the analysis, whether it be spectral or statistical, will be passed on to a cognizant design or test engineer.

2. VIBRATION DATA ANALYSIS

Vibration data may be measured, reduced, and presented in two different ways. One method is to consider the magnitude of the vibration at a given frequency of vibration. In other words construct a plot of magnitude versus frequency. The other type of presentation is one of magnitude of vibration at a given frequency over a period of time, i.e. magnitude versus time plots as shown in Figure 1.

A plot of both time and frequency versus magnitude is shown in Figure 2, and illustrates the difference between the two.

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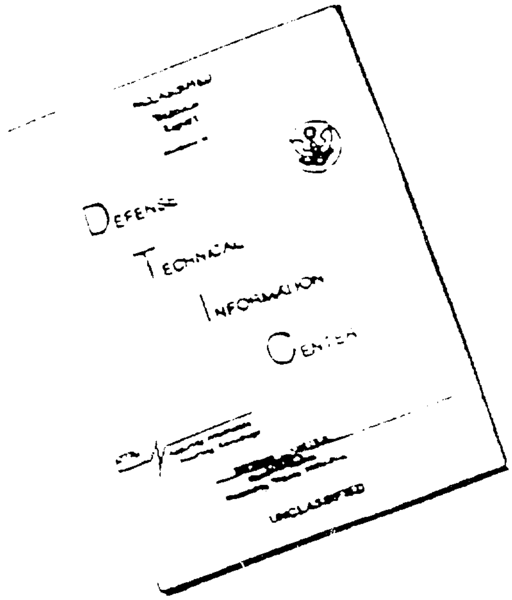

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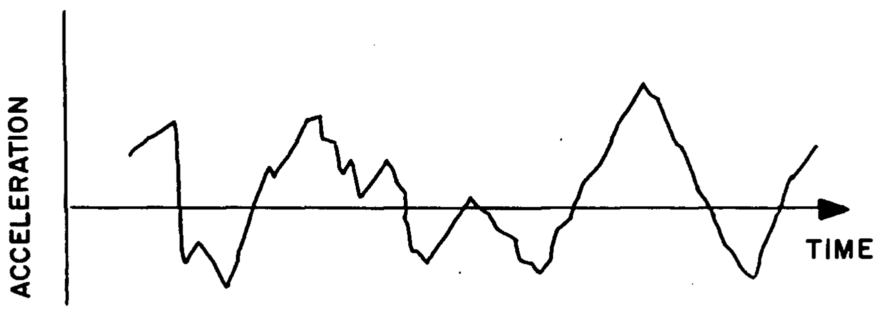


Figure 1. Typical Vibration Time-History Broadband Random Vibration.

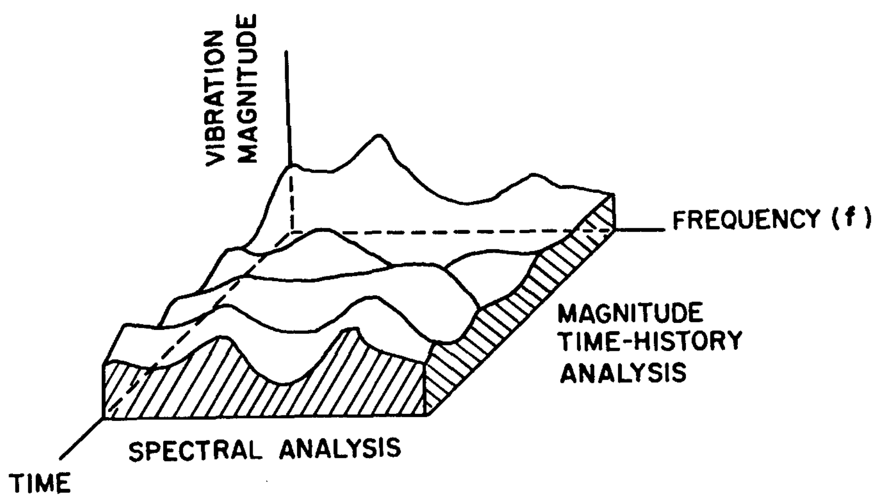
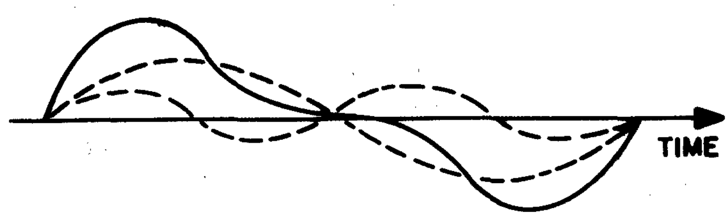


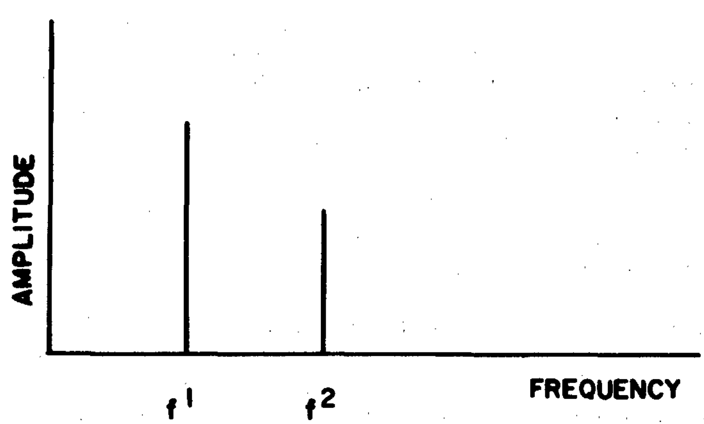
Figure 2. Vibration Magnitude Surface.

2.1 TYPES OF FREQUENCY-MAGNITUDE PLOTS

A presentation of the magnitude of a vibration versus the frequency of a vibration is called a frequency spectrum. The analysis of a periodic wave, as shown in A of Figure 3 when performed over a large finite number of cycles, will produce a line spectrum or discrete frequency spectrum as shown in B of Figure 3. Instead of a graphical presentation, the results of this type of analysis might well be presented as a tabulation of frequencies and associated amplitudes.



A. Typical Vibration Time-History Sum of Two Sinusoids



B. Results of Data Analysis of Time-History of A., Sum of Two Sinusoids

Figure 3. Typical Vibration Time-History and Data Analysis.

This type of presentation might represent vibration magnitudes through fixed filter bandwidths at an instant in time. For example, this

could be accomplished by passing the instantaneous acceleration signal through a set of parallel filters and recording (on an oscillograph) each filter output simultaneously. The data required for the frequency spectrum at time t_1, would then be taken by observing on each oscillograph trace the amplitude present at t_1. (See Figure 4.)

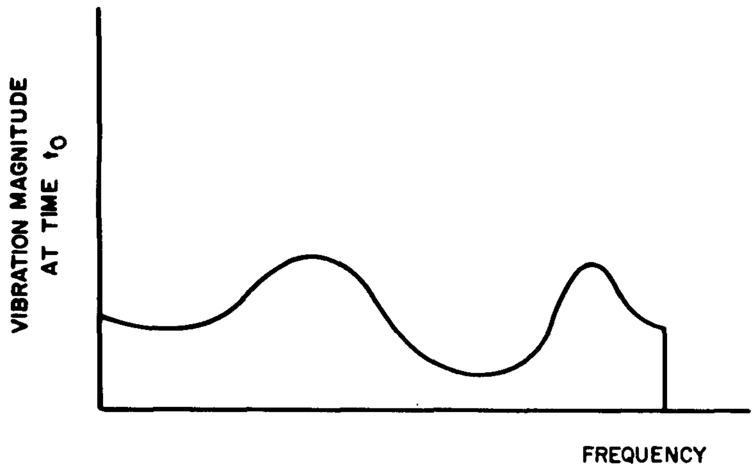


Figure 4. Typical Plot of Spectral Analysis.

Sometimes it is desired that the amplitude associated with the frequency spectrum be a time-averaged value. For example, it might be desirable to plot the root mean square (rms) acceleration experienced at a particular station of a missile during the free flight portion of a given missile maneuver. In this case, it would be necessary to average the instantaneous acceleration in some way, over the time of interest. In some cases, obtaining data related to an instant in time can be precluded by averaging time constants and damping built into the instrumentation being used.

Although the presentation form discussed previously, which involves frequency as the abcissa, can be referred to as "spectral analysis", the concept most often associated with this term is that of the "power spectrum".

In general, vibration data are aperiodic (i.e., random in nature). Such a vibration signal cannot be resolved into discrete sinusoidal components such as amplitude and phase spectra. However, the energy associated with a random time function taken over a finite length of time is finite. Thus, many data can be obtained for an arbitrary signal, if the distribution of its power with respect to frequency is known.

Actually, it is the rate of change of average power with frequency which is of prime interest and this introduces a presentation concept of "power spectral density".

2.2 MATHEMATICAL RELATIONSHIPS THAT DESCRIBE VIBRATION PHENOMENA

2.2.1 Sinusoidal Functions

The simplest time-varying quantity of concern in vibration is a sinusoidal function of constant peak amplitude, Am and frequency, f.

$$a(t) = Am \sin(\omega t + \phi) \tag{1}$$

where $\omega = 2\pi f$

ϕ = phase angle with respect to the time origin.

2.2.2 Periodic Functions

A function which has a waveform which repeats itself exactly during each time period τ such that:

$$a(t) = a(t \pm n\tau) \tag{2}$$

where n is any integer, then it is said to be a periodic function having a fundamental frequency where f equals $1/\tau$. A periodic function may be represented by the line spectrum as shown in Figure 5.

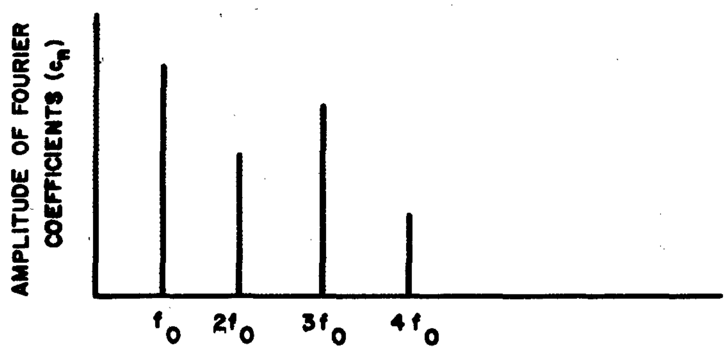


Figure 5. Line Spectrum to Represent Fourier Series of Periodic Function. Fundamental Frequency $f_0 = 1/\tau$, where τ is Period of Function.

2.2.3 Phase - Coherent Functions

A phase-coherent function is defined as a sine wave whose amplitude and frequency vary smoothly with time. It is assumed that the changes in amplitude and frequency are significant in magnitude only over a time duration equal to a number of cycles of the waveform. It is expressed as:

$$a(t) = a_0(t) \sin[\omega(t) + \theta] \tag{3}$$

where a_0 and ω are functions of time and θ is the phase angle of the waveform

at the origin.

In a strict sense, phase-coherence is determined by a single reference frequency which persists for all time. This type of function is useful in describing a vibration measured in a system whose material frequency varies with time.

2.2.4 Complex Functions

A complex function is a time-history which consists of a sum of sinusoidal functions but which is not necessarily periodic. It can be represented graphically by a line spectrum such as that shown in Figure 5. The lines are not equally spaced along the frequency axis for a complex function, and if the function consists of a continuous distribution of sinusoidal functions, it is best described by the Fourier spectrum.

2.2.5 Random Functions

A random time function is composed of a distribution of sine waves which is continuous and contains all frequencies. It cannot be represented in a deterministic sense; that is, if its value at a given instant of time is known, it is known only that at some time Δ t seconds later there is a certain probability that the value will be within a specific range of values. Random functions can only be described in statistical terms as opposed to a sine wave which is completely deterministic.

2.2.6 Transient Function

A transient function is a time-varying function which is nonzero only over a given restricted time interval. Shock is a force or acceleration represented by a discontinuous function of time. Physically, no force or acceleration is strictly discontinuous and, therefore, shock also refers to those forces and accelerations that have first time derivatives of very high magnitude and are not sinusoidal.

2.2.7 Fundamental Statistical Formulas

a. Mean Value. The mean value of a function x(t) is defined by:

$$\overline{x} = \frac{1}{T}\int_0^T x(t)dt \qquad (4)$$

Where T is the time interval over which the data points x(t) are taken. In most vibration cases the mean value will be zero.

b. Average value. The term "average value" of a function x(t) is employed to denote the time average of the magnitude of a function with zero mean value:

$$|\bar{x}| = \frac{1}{T}\int_{0}^{T} |x(t)|\ dt \qquad (5)$$

c. Mean square value. The mean square value of a function $x(t)$ is the time average of the square of the function:

$$|\bar{x}| = \frac{1}{T}\int_{0}^{T} |x(t)|^2\ dt \qquad (6)$$

d. Root mean square value (RMS). The RMS value is the square root of the mean square value.

2.3 Analysis of Periodic Functions

2.3.1 Fourier Series

Any periodic function $f(\theta)$, with period 2π, which satisfies the Dirichlet conditions may be expanded into a Fourier series

$$f(\theta) = \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \qquad (7)$$

which consists of a sum of sine waves whose frequencies are all multiples of the fundamental frequencies. The Dirichlet conditions to be met by the function are:

a. It must have, at most, a finite number of discontinuities in one period.

b. It must have, at most, a finite number of maxima and minima in one period.

c. The integral $\int_{-\pi}^{\pi} |f(\theta)|\ d\theta$ must be finite.

The coefficients a_n and b_n are given by:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \cos n\theta\ d\theta \qquad (8)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \sin n\theta\ d\theta \qquad (9)$$

The series can be expressed alternatively as:

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} C_n \cos (n\omega t - \psi n) \qquad (10)$$

where $C_n = \sqrt{a_n^2 + b_n^2}$ = the Fourier coefficients (11)

$\psi n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$ = the relative phase (12)

In each case, $\frac{a_o}{2}$ is equal to the average, or direct current (d-c), value of the function under study. A plot of $|C_n|$ versus $n\omega$ such as shown in Figure 6 is called the frequency spectrum of the given f(t). For a periodic function, this graph is a discrete frequency spectrum, as opposed to a continuous spectrum such as arises in the study of aperiodic signals.

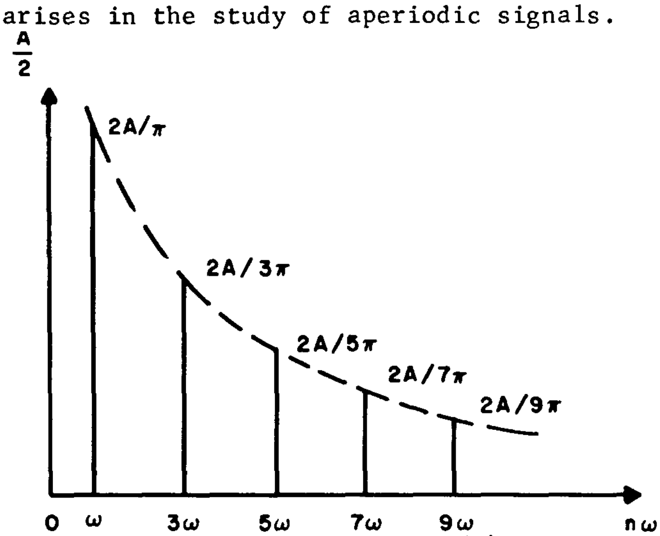


Figure 6. Frequency Spectrum of f(t) in Figure 7.

If the given periodic function exhibits certain symmetry conditions, several simplifications can be applied to the analysis.

If the function is even, such that

$$f(t) = f(-t), \qquad (13)$$

then the Fourier series expansion contains only cosine terms ($b_n = 0$) and the constant term (if $a_o \neq 0$).

If the function is odd, such that

$$f(t) = -f(-t), \qquad (14)$$

then the Fourier series expansion contains only sine terms ($a_n = 0$).

The periodic function $f(\theta)$ with period 2π contains only even harmonics if it satisfies the condition that:

$$f(\theta \pm \pi) = f(\theta) \tag{15}$$

Likewise, the periodic function contains only odd harmonics, if it satisfies the condition that

$$f(\theta \pm \pi) = -f(\theta) \tag{16}$$

For example: Find the Fourier series expansion of the periodic rectangular wave of Figure 7:

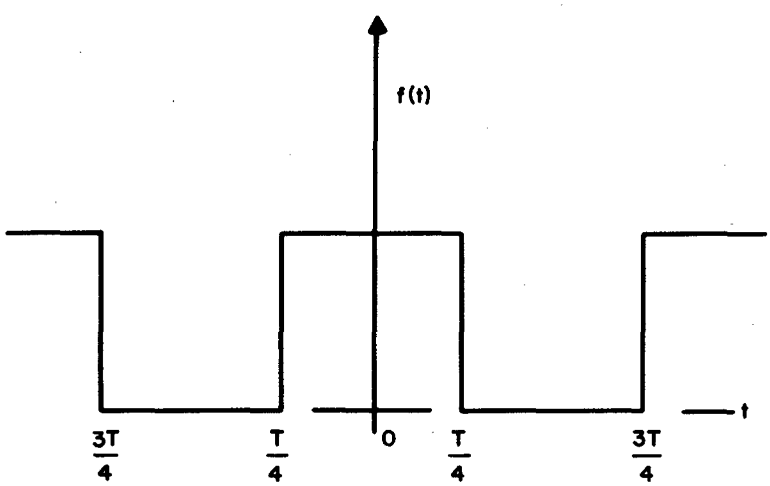


Figure 7. A Periodic Rectangular Wave.

which is expressed over one period $\left(-\frac{T}{2} < t < \frac{T}{2}\right)$ as:

$$f(t) = \begin{cases} 0 & \frac{T}{2} < t < \frac{T}{4} \\ A & \frac{T}{4} < t < \frac{T}{4} \\ 0 & \frac{T}{4} < t < \frac{T}{2} \end{cases} \tag{17}$$

Using equation (8), we determine the a_n coefficient for cosine terms as:

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos n\omega t \, dt \qquad (18)$$

$$= \frac{2}{T} \int_{-\frac{T}{4}}^{\frac{T}{4}} A \cos n\omega t \, dt \qquad (19)$$

$$= \frac{2A}{n\omega T} \sin n\omega t \Bigg|_{-\frac{T}{4}}^{\frac{T}{4}} \qquad (20)$$

$$a_n = \frac{2A}{n\pi} \sin \left(\frac{n\pi}{2}\right) \qquad (21)$$

By inspection of the even symmetry of the given wave, it can be seen that the $b_n = 0$ for this case.

The average value can be determined by substitution of $n = 0$ in $A_n = 2A/n\pi \sin (n\pi/2)$, which then takes the indeterminate form o/o. This can be evaluated using L'Hospital's rule or the average value in this case can be determined to be $\frac{A}{2}$. The total series can then be expressed as:

$$f(t) = \frac{A}{2} + \frac{2A}{\pi} (\cos \omega t - \frac{1}{3} \cos 3\omega t + \frac{1}{5} \cos 5\omega t - \ldots.) \qquad (22)$$

Fourier series can also be written in exponential form as:

$$f(t) = \sum_{n=-\infty}^{+\infty} \alpha_n \epsilon^{jn\omega t} \qquad (23)$$

where

$$\alpha_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \epsilon^{-jn\omega t} dt$$

with n ranging over all values from $-\infty$ to $+\infty$.

An advantage of the exponential form of the Fourier series is that is provides a more convenient transition to Fourier integrals and transforms.

2.3.2 Harmonic Analysis

Harmonic Analysis is the determination of the coefficients of the series expansion for periodic waves given in the previous section. One method of harmonic analysis uses parallel bandpass filters. The recorded signal is played back through each filter in turn, and the filter output is again recorded for further analysis on an instrument such as a level recorder or an oscillograph. Another method widely used takes advantage of the "heterodyning" effects of a variable oscillator as it is mixed with the input data, and then filtered. The filtering is performed at either a fixed intermediate frequency or at a zero IF. The local oscillator is made to sweep automatically at some preselected rate. A tape loop input is always used with this type of analyzer, whereas, in the parallel filter concept, a simple tape input scheme is acceptable.

2.3.3 Statistical Analysis

Statistical analysis of shock and vibration data is utilized to determine the "amplitude spectrum" of stochastic or random functions. It largely concerns the predetection or probability that the vibration data will fall between arbitrary levels. From this analysis a distribution function is usually derived that will permit an evaluation of vibration and shock effects.

2.3.3.1 Density Functions

In general, it is desirable to produce, in graphic form, a presentation from which can be obtained a discrete probability associated with a certain event. (For example, the probability of occurrence of a certain range of accelerations.) One concept which can be introduced to do this is that of the probability density function.

By way of approaching the concept of probability density, assume that the distribution of height among American males must be determined. The height of a man may take on any value within a specified interval and thus represents a continuous random variable. (As also does most inflight vibration data.) Assume that a representative sample of 1000 men is selected for this determination. Then, their heights are measured and they are grouped according to the nearest even inch (5 feet, 0 inches; 5 feet, 2 inches; 5 feet, 4 inches; etc.). All heights from 4 feet, 11 inches, to 5 feet, 1 inch, will, for example, be grouped in the 5-foot, 0-inch category. The relative frequency of men found in each height interval is then the number grouped in that interval divided by the total number (1000). A typical plot of such a height distribution is shown in Figure 8. In Figure 8, x represents the height, n_x/n the relative number of men grouped in an interval Δx inches about x as the mean. n is the 1000-man sample. Since all heights in a 2-inch interval about the even heights are grouped together, they are shown as horizontal lines covering the 2-inch grouping. For a large enough sample, it may be stated that n_x/n represents the probability $P(x_j)$, that

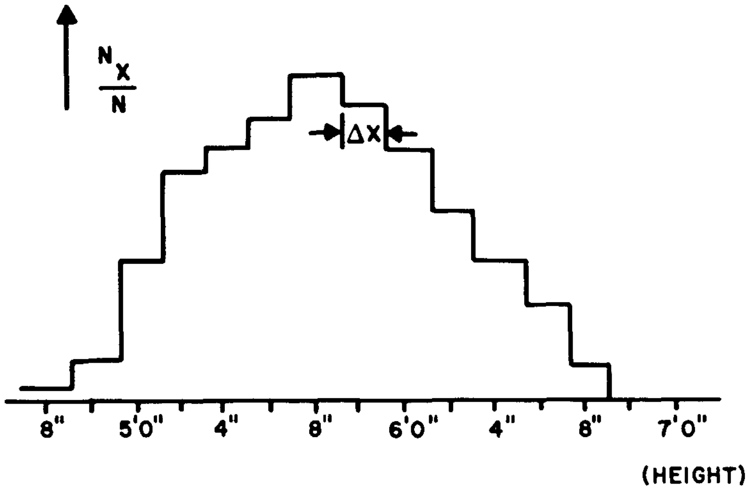


Figure 8. Relative Distribution of Heights.

the height of an American male will be between:

$$x_j - \frac{\Delta x}{2} \text{ and } x_j + \frac{\Delta x}{2} \tag{24}$$

To do away with the dependence of the size of the interval Δx chosen for the height distribution, the height distribution is represented at a particular height and of interval Δx by a rectangle of area n_x/n. The height of this rectangle will be $\left(\frac{1}{\Delta x}\right)\left(\frac{n_x}{n}\right)$, its width will be Δx.

Such a relative-frequency curve is called a histogram. Two histograms for the height example used here are shown in Figure 9 (where Δx equals two inches) and Figure 10 (where Δx equals one inch).

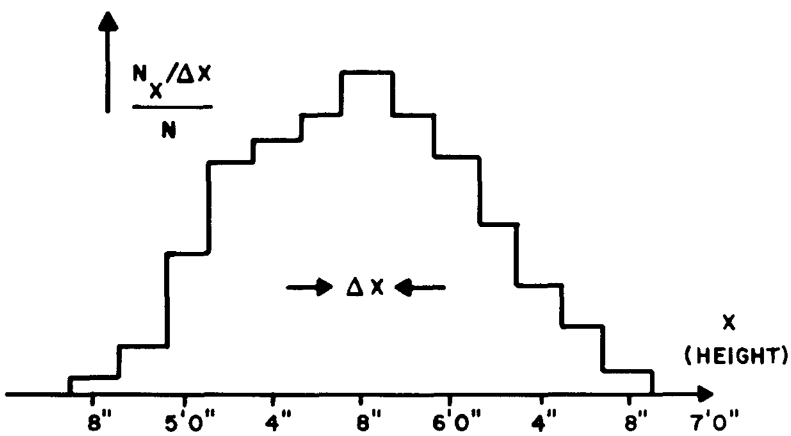


Figure 9. Histogram of Height Distribution, Δx Equals Two Inches

For a large enough sample size, the probability that a man's height will lie between two given heights, (e.g., 3 feet 7 inches, and 5 feet, 9 inches) will now be the area of the histogram between these two limits.

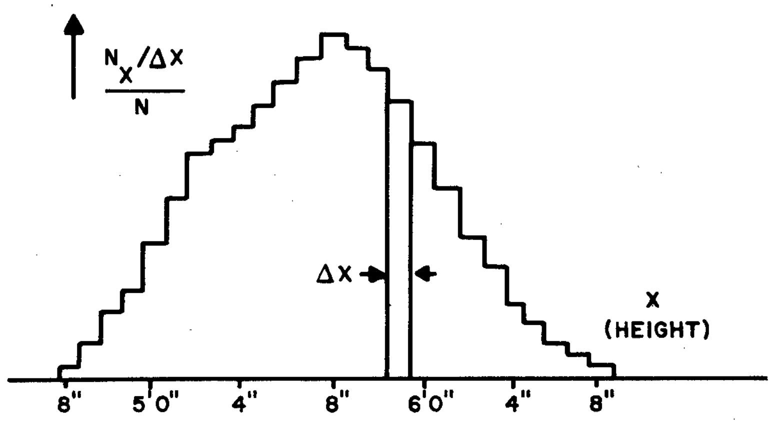


Figure 10. Histogram of Height Distribution, Δx Equals One Inch.

If the histogram approaches a smooth curve as Δx approaches zero, the ordinate takes on the form of a probability-density function, with the area under the curve between two points giving the probability that the height will be found between these two points. The probability-density function as the limiting case of the two histograms of Figures 9 and 10, is shown in Figure 11.

The area between the two points, 5 feet, 6 inches, and 5 feet, 8 inches, shown crosshatched in the figure, represents the probability that a man's height will be found in that range.

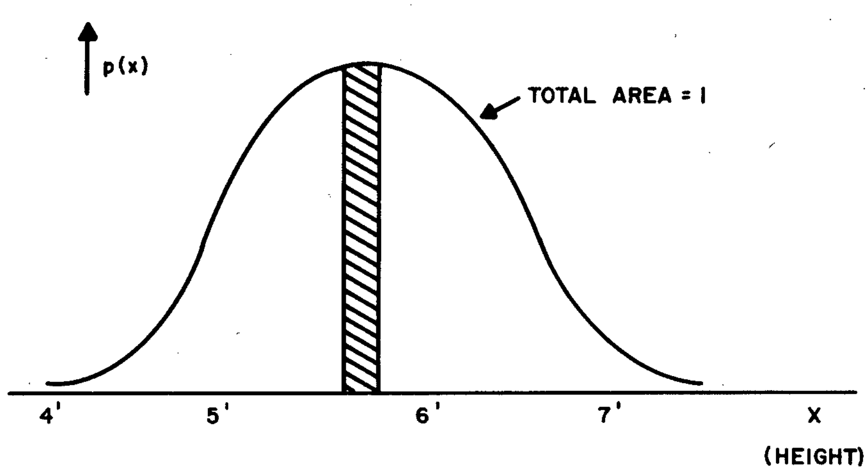


Figure 11. Probability-Density Function Corresponding to Height Histogram.

Thus, what has been established then, is a function p(x) where:

$$p(x) = \lim_{\substack{\Delta x \to 0 \\ n \to \infty}} \frac{nx/\Delta x}{n} \tag{25}$$

where p(x) is the probability-density function. Px represents the number of samples of the total n found in the range x - Δ x/2 to x + Δx/2. To then find the probability that a value is in a specified range, x_1 to x_2, p(x) is integrated through this range by:

$$\text{Prob } (x_1 < x < x_2) = \int_{x_1}^{x_2} p(x)\, dx \tag{26}$$

Since it is certain that every measurement must yield some real value, it is necessary to have:

$$\int_{-\infty}^{+\infty} p(x)\, dx = 1 \tag{27}$$

The histogram or probability-density curves must be normalized to have unity area.

To determine the statistical averages from the probability-density function just described, it can be reasoned that in the limit, as the number of measured values becomes large and as Δx approaches 0, then:

$$\text{av } F(x) = \int_{-\infty}^{\infty} F(x)\, p(x)\, dx \tag{28}$$

where the abbreviation "av F(x)" is used to mean the average value of F(x) approached by a large number of trials. Other designations in use include "$< F(x) >$" and "mathematical expectation of F(x), written E[F(x)].

As a special case, the average value of x is:

$$\text{av } x = \int_{-\infty}^{\infty} x\, p(x)\, dx \tag{29}$$

The average value of x, av x, is also called the first moment, m_1, of x by analogy with the concept of moments in mechanics. In mechanics, the first

moment of a group of masses is just the average location of the masses or their center of gravity. For example, the center of gravity of the masses m_1, m_2 ... m_5, in Figure 12 can be found simply by taking amounts about the point where x equals 0. The average value of x, or the center of gravity, is:

$$\text{av } x = \frac{m_1 \ x_1 + m_2 \ x_2 + \ldots}{m_1 + m_2 \ \ldots} \tag{30}$$

If the mass is not concentrated as discrete points, but is continuous over the bar, the center of gravity can be calculated by considering a differential mass m(x) dx [with m(x) the linear mass density] located x units from the origin. The center of gravity is then found by summing all the mass contributions. This sum becomes an integral to yield:

$$m_1 = \text{av } x = \int_0^1 x m(x) \ dx \tag{31}$$

The second moment in mechanics is just the moment of inertia of a mass or the turning moment of a torque about a specified point. By analogy with mechanics the second moment of a random variable is the average value of the square of the variable. For a continuous variable, this is given by:

$$m_2 = \text{av } x^2 = \int_{-\infty}^{\infty} x^2 \ p(x) \ dx \tag{32}$$

Higher moments can be defined, but in general, the nth moment of x or av x^n is given by:

$$m_n = \text{av } x^n = \int_{-\infty}^{\infty} x^n \ p(x) \ dx \tag{33}$$

The density function p(x) plays the role of a weighting function throughout.

One significance of these moments in voltage measurements for instance is that m_1 gives the d-c value whereas, m_2, is found to give the mean squared voltage or the mean power.

2.3.3.2 Distribution Functions

A second important function called the distribution function p(x), is defined as the probability that the value is less than some specified x. It is given by:

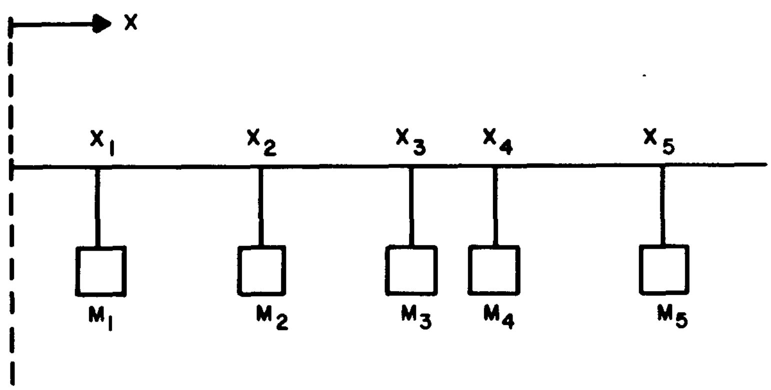
$$p(x) = \int_{-\infty}^{x} p(x)\, dx \qquad (34)$$

From the non-negative character of the density function p(x), it can be seen that p(x) cannot decrease with increasing x and also that

$$p(-\infty) = 0 \quad \text{and} \quad p(\infty) = 1$$

p(x) is a continuously or monotonically increasing function going from 0 to 1 as shown in Figure 13. At any point where the random variable has a finite probability the function p(x) has a discontinuity equal to the amount of the respective probability. The value of the distribution function

$p\left(\frac{x >}{\sigma}\right)$ at the minimum possible value of x is $p\left(\frac{x\ min}{\sigma}\right) = 1.0$ since x is never less than x min; for the maximum possible value of x, $p\left(\frac{x\ max.}{\sigma}\right)$ is zero since x never exceeds x max. Since the probability of exceeding that value is one minus the probability of not exceeding that value, the ordinate of Figure 19 may be changed to $p\left(\frac{x <}{\sigma}\right)$ if the scale is changed to vary from 1 to 0 instead of from 0 to 1.



$M_1 + M_2 + M_3 + M_4 + M_5 = 1$

Figure 12. Discrete Mass Distribution.

The term σ (sigma) represents the standard deviation of the random variable from the mean value of the frequency function. It is used as a measure of the spread of the density function about the mean value, m_1 and is the square root of the variance or second central moment μ_2.

2.3.3.3 Comparison of Sinusoidal and Random Functions

Although a sinusoidal function is deterministic, the probability density and probability distribution of a sine wave may be determined for comparison with those of a random function. The comparison may be made on the basis of (1), the instantaneous values $x|t|$ of the function, or (2) the peak values or maxima $x_p(t)$ of the function. It will be seen in the following paragraphs that although the instantaneous values of a truly random wave can approach infinity, the probability density function of the random wave under consideration is well behaved throughout the entire region of observation. On the other hand, although the instantaneous values of a sine wave are well behaved and in fact, deterministic, the probability density function becomes asymptotic to infinity at the peak amplitudes of the sine wave. An investigation would reveal that although the distribution function in all cases is well behaved, several probability density functions display discontinuities. In general, as the crest factor (ratio of peak to rms value) of a signal decreases, the probability density function begins to increase indicating the occurrence of one or more discontinuities.

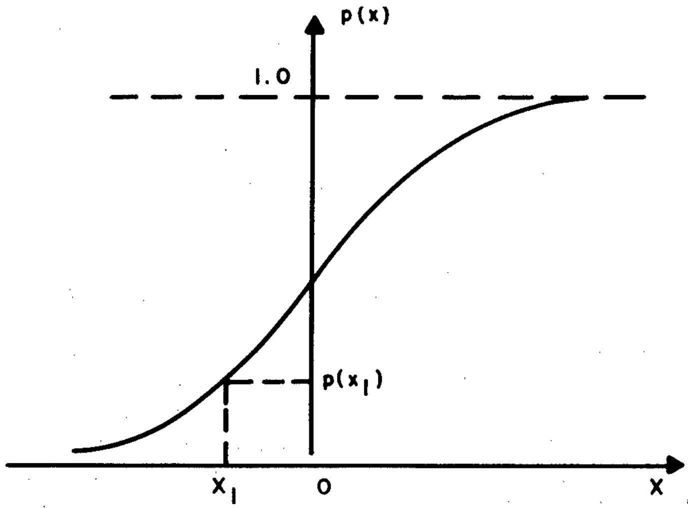


Figure 13. Probability-Distribution Function.

2.3.3.4 Distribution of Instantaneous Values

A comparison of some of the identifying characteristics of the probability densities and distributions for the instantaneous values of a sinusoidal function and the particular case of Gaussian random vibration is shown in Figures 14 through 17. The term Gaussian is used to describe a random function whose instantaneous value is defined by a probability density which is proportional to the exponential of a negative quadratic function of the values of the variable. It is one of the most important distributions in noise theory and is expressed for one variable as:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \epsilon^{-(x-x_o)^2/2\sigma^2} \tag{35}$$

Here the parameters have been adjusted to give the required condition:

$$\int_{-\infty}^{\infty} p(x)\, dx = 1 \tag{36}$$

and also:

$$m_1 = \int_{-\infty}^{\infty} x\, p(x)\, dx = x_o \tag{37}$$

$$m_2 = \int_{-\infty}^{\infty} x^2\, p(x)\, dx = x_o^2 + \sigma^2 \tag{38}$$

$$\mu_2 = \sigma^2 = \int_{-\infty}^{\infty} (x-m_1)^2\, p(x)\, dx = m_2 - m_1^2 \tag{39}$$

That is, x_o is the mean, σ^2 is the variance, and σ is the standard deviation.

An important theorem in statistics called the central limit theorem shows under general assumptions that the distribution of the sum of an indefinitely large number of other independently distributed quantities must approach the Gaussian distribution no matter what the individual distributions may be. In other words, if x_1, x_2, ... x_n, are independent random variables with arbitrary distributions, the sum:

$$x = \frac{x_1 + x_2 + \ldots x_n}{\sqrt{n}} \tag{40}$$

approaches a normally distributed variable as n approaches infinity. If x_1 has mean zero and variance of $\sigma_1^2 < \infty$, then x has mean zero and variance σ^2 where:

$$\sigma^2 = \frac{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2}{n} \tag{41}$$

The probability density function of the instantaneous values of a random variable following the normal or Gaussian distribution is shown by curve A in Figure 18. To express the probability density function of the instantaneous value of a sinusoid, several ideas in functional relationships must be introduced.

In general, if the probability density function $q(\theta)$, of a variable θ is available and it is desirable to calculate the probability density function $p(x)$ for x equals $F(\theta)$, it is convenient to invert the functional relationship and write $\theta = f(x)$. Then by direct substitution:

$$q(\theta)d\theta = q[f(x)]f'(x)\ dx \qquad (42)$$

where $f'(x) = \frac{d\ f(x)}{dx}$.

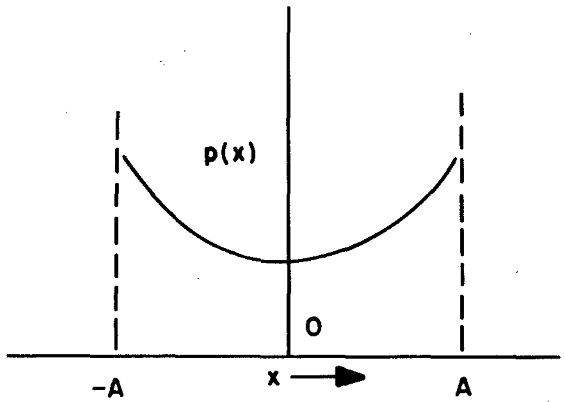


Figure 14. Probability Density Function for Sinusoidal Distribution.

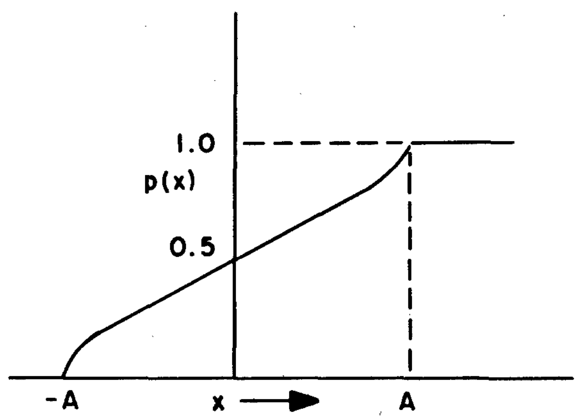


Figure 15. Distribution Function for Sinusoidal Case.

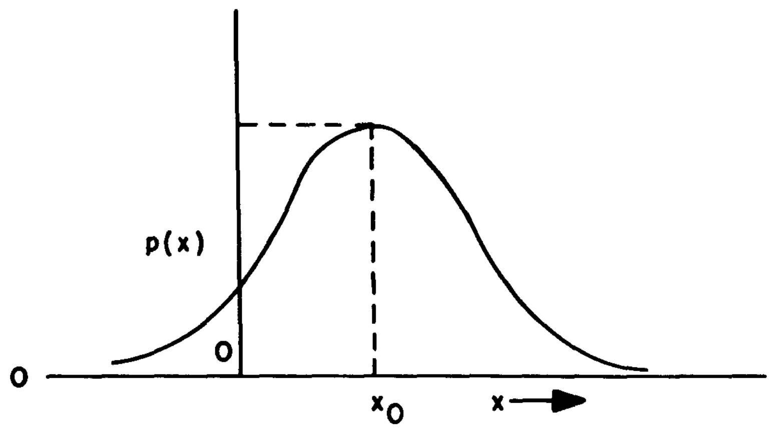


Figure 16. Probability Density Function for Gaussian Distribution.

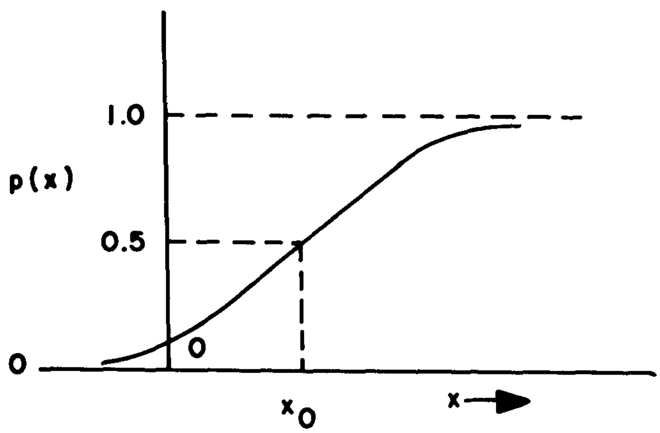
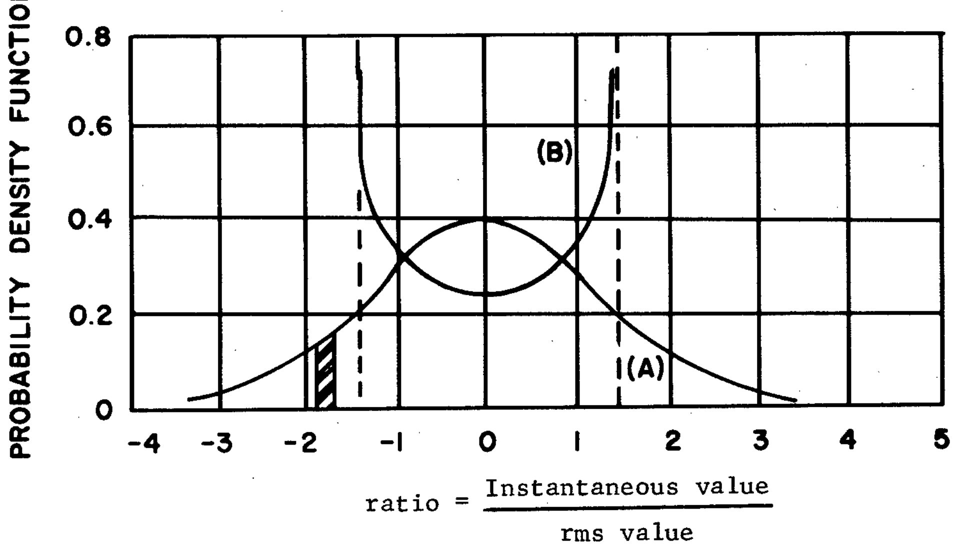


Figure 17. Distribution Function for Gaussian Case.



(A) Gaussian or normal distribution.
(B) Distribution of instantaneous values of a sine wave.

Figure 18. Normalized Probability Density Functions.

Hence if $f'(x)$ is a single-valued function of x, then:

$$p(x) = q[f(x)]\,|f'(x)| \tag{43}$$

If $f(x)$ is multiple valued, the expression on the right is summed over all the values. For the sinusoidal case let

$$x = A \sin \theta \tag{44}$$

where θ has a uniform distribution in range $\frac{-\pi}{2}$ to $\frac{3\pi}{2}$. Then the values of x are distributed like those of the ordinates of a sine wave with amplitude A. To find the probability density function for x:

$$q(\theta) = \begin{cases} 0 \text{ when } \theta < \frac{-\pi}{2} \text{ and } \theta > \frac{3\pi}{2} \\ \frac{1}{2\pi} \text{ when } \frac{-\pi}{2} < \theta < \frac{3\pi}{2} \end{cases} \tag{45}$$

where $\frac{1}{2\pi}$ ensures a normalized condition.

Then solving equation (44) for θ as a function of x:

$$\theta = f(x) = \arcsin \frac{x}{A} \tag{46}$$

$$f'(x) = \frac{d[f(x)]}{dx} = [A^2 - x^2]^{-\frac{1}{2}} \tag{47}$$

Note that each value of x in the range -A to A corresponds to two values of θ and hence that the right side of equation (43) must be multiplied by two to obtain $p(x)$. Then:

$$p(x) = \begin{cases} 0 \text{ when } x < -A \text{ and } x > A \\ \frac{(A^2 - x^2)^{-\frac{1}{2}}}{\pi} \text{ when } -A < x < A \end{cases} \tag{48}$$

A plot of $p(x) = \dfrac{1}{\sqrt{A^2 - x^2}\,\pi}$ is shown by curve B in Figure 18 and also in Figure 14. Note that at $x = \pm A$, $p(x)$ blows up. The average value is given by:

$$m_1 = \int_{-A}^{A} \frac{x\,dx}{\pi\sqrt{A^2 - x^2}} = 0 \tag{49}$$

and the mean square is:

$$m_2 = \int_{-A}^{A} \frac{x^2\,dx}{\pi\sqrt{A^2 - x^2}} = \frac{A^2}{2} \tag{50}$$

The mean square value could also have been obtained from the original function given that it was periodic with period 2π.

$$|\overline{x}|^2 = \frac{1}{T}\int_{0}^{T} |x(t)|^2\,dt \tag{51}$$

$$|\overline{x}|^2 = \frac{1}{2\pi}\int_{0}^{2\pi} |A \sin\theta|^2\,d\theta = \frac{A^2}{2} \tag{52}$$

Likewise the distribution function is:

$$p(x) = \int_{-A}^{x} \frac{dx}{\pi\sqrt{A^2 - x^2}} = \frac{1}{\pi}\left(\frac{\pi}{2} + \text{arc sin}\,\frac{x}{A}\right) \tag{53}$$

Thus, it can be seen that it is possible for the probability density to become infinite provided the integral over a finite interval, including the singularity, exists.

The probability density functions of equations (35) and (48) are symmetrical about their means. If the mean is assumed to be zero, then the probability that x exceeds a given absolute value (or magnitude) $|x|$ is twice the probability that it exceeds the same absolute value in either the positive or negative sense. Therefore, it is convenient to plot the probability distribution function in terms of the absolute value of x, i.e.,

$$p\ (|x| \geq) \quad \text{or} \quad p\left(\frac{|x|}{\sigma} \geq\right)$$

as shown in Figure 19.

The notation $\frac{x}{\sigma}$ or $\frac{x}{\sigma}$ is often used for the abscissa value in place of x or $|x|$. Division by the standard deviation in a sense "normalizes" the plot in as much as it reduces the abscissa value to a dimensionless quantity which holds its significance no matter what the value of the random variable. The probability distribution functions for the Gaussian and sinusoidal functions are obtained by integration of equations (35) and (48), and are:

Gaussian
$$p\left(\frac{|x|}{\sigma} \geq\right) = \frac{2}{\sqrt{2\pi}} \int_{\frac{x}{\sigma}}^{\infty} \epsilon^{-(x_1{}^2/2\sigma^2)}\, d\left(\frac{x_1}{\sigma}\right) \tag{54}$$

where x_1 is a dummy variable.

Sinusoid
$$p\left(\frac{|x|}{\sigma} \geq\right) = \frac{2}{\pi} \text{arc cos } \frac{x}{\sigma\sqrt{2}} \tag{55}$$

The probability associated with the Gaussian distribution function above is essentially twice the area under either side of the normal curve from the point in question out to the extreme value of the respective side. It could also be expressed as

$$p\left(\frac{x}{\sigma} \geq\right) = 1-2\left[\int_{0}^{x/\sigma} \epsilon\left(\frac{1}{\sqrt{2\pi}} -x_1{}^2/2\sigma^2\right) d\left(\frac{x_1}{\sigma}\right)\right] \tag{56}$$

The probability that $\frac{x}{\sigma}$ is greater than a given normalized value is written "$p\left(\frac{x}{\sigma} \geq\right)$", conversely the probability that $\frac{x}{\sigma}$ is less than a given normalized value is written "$p\left(\frac{x}{\sigma} \leq\right)$". Also, $p\left(\frac{x}{\sigma} \geq\right) = 1 - p\left(\frac{x}{\sigma} \geq\right)$. The probability that an instantaneous voltage which follows the normal distribution will be below some value $K\sigma$ (where K is a constant) can be expressed in terms of the error function as:

$$\text{Prob}\quad (|x| < K\sigma) = \frac{K}{\sqrt{2}} \tag{57}$$

The error function is described mathematically as:

$$\text{erf } x = \frac{2}{\sqrt{\pi}} \int_0^x \epsilon^{-y^2} dy \tag{58}$$

and is tabulated in many books on statistics and in mathematical tables.

The cumulative probability distribution can be related to the error function as follows:

$$p(x) = \tfrac{1}{2} \left[1 + \text{erf } \frac{(x-a)}{\sigma\sqrt{2}} \right] \tag{59}$$

where p(x) is the cumulative distribution and a is the mean of the normal distribution under study.

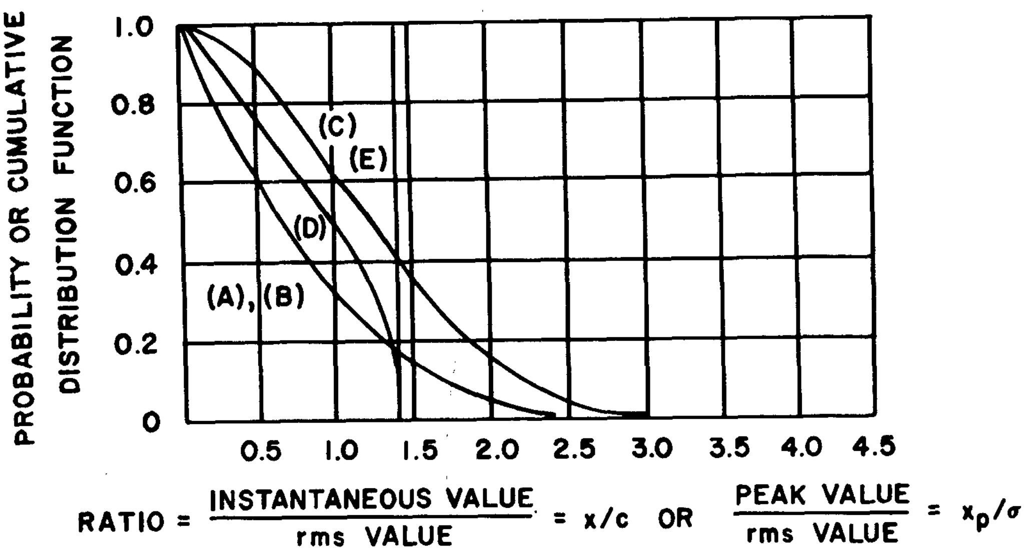
2.3.3.5 Distribution of Peak Values

When the peak values or maxima of a function are considered, the two statistical functions differ from those found for the instantaneous values. For a sine wave, all maxima are of equal magnitude and the probability density function $p(x_p/\sigma)$ becomes a Dirac delta function as shown by curve B in Figure 20. For broadband Gaussian vibration, i.e., vibration with non-zero spectral density over a frequency bandwidth which is not small compared to the average or center frequency of the bandwidth, the distribution of peak values is normal as shown by curves A in Figure 18 and by the curve in Figure 16. However, for narrow band Gaussian noise, i.e., noise with negligible spectral density except in a frequency bandwidth which is small compared to the center frequency, the distribution of peak values becomes the Rayleigh distribution. The probability density and distribution functions for the Rayleigh distribution are defined by:

a. Probability density,

$$p\left(\frac{|x_p|}{\sigma}\right) = \frac{x_p}{\sigma}\, \epsilon^{-x^2{}_p/2\sigma^2} \tag{60}$$

b. Probability distribution,



(A) Instantaneous values of broad band and narrow band random vibration - Gaussian distribution (Equation 54);
(B) Peaks of broad band random vibration (Equation 54);
(C) Peaks of narrow band random vibration (Equation 61);
(D) Instantaneous values of a sine wave (Equation 55);
(E) Peak values of a sine wave.

Figure 19. Probability Distribution Functions.

$$p\left(\frac{|x_p|}{\sigma}\right) = \epsilon^{-x^2 p/2\sigma^2} \tag{61}$$

Since the Rayleigh distribution has only positive values, the density function can also be written as:

$$P(x_p) = \frac{x_p}{\sigma^2}\,\epsilon^{-x^2 p/2\sigma^2} \tag{62}$$

The relations given by equations (60) and (61) are shown graphically by curve A of Figure 20 and curve C of Figure 19 respectively. Figures 18 through 20 are plotted on the basis that the mean value of the function x(t) is zero. In the more general case of nonzero mean value, the absicssae of these figures are $\left(\frac{x - x}{\sigma}\right)$ where x is the mean value of the function x(t). However, in vibration data analysis, it is common practice to constrain the mean value to zero by use of a high-pass filter prior to statistical analysis (elimination of d-c mean). Thus, these figures show the form usually encountered. Figures 14 through 20 show that the probability density and probability distribution curves for sinusoidal and random functions differ considerably, thus providing identifying characteristics for each type of function. When the time-history x(t) is a combination of sinusoidal and random functions, the shape of the probability density and distribution curves depends on the relative magnitudes of each type of function.

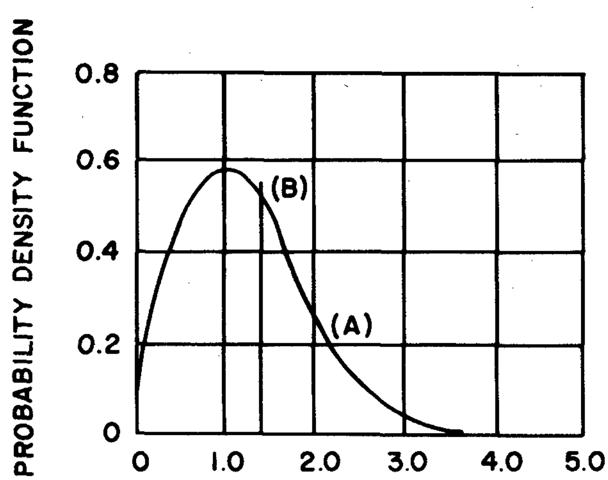


Figure 20. Normalized Probability Density Functions.

2.3.3.6 Gaussian Distribution

The Gaussian or normal distribution mentioned earlier is described by the density function:

$$p(x) = \frac{\epsilon^{-(x-a)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \tag{63}$$

When plotted, it has the characteristic bell-shaped curve of Figure 21.

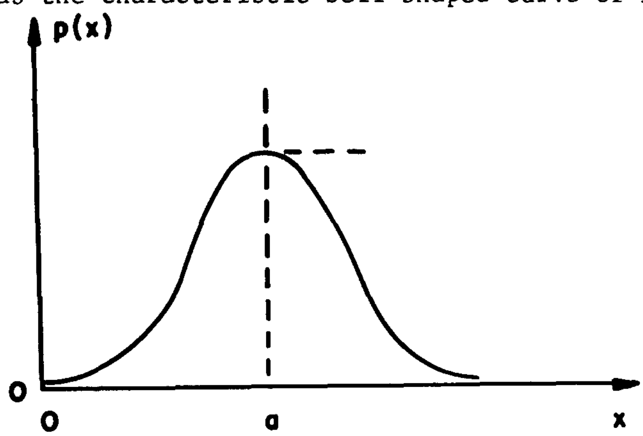


Figure 21. Density Function, Gaussian Distribution.

The curve is symmetrical about the point where x equals a and has a width proportional to σ. It can be shown that the mean value of the distribution above is truly a, by solving for the first moment. Thus:

$$m_1 = \int_{-\infty}^{+\infty} x\ p(x)\ dx \tag{64}$$

$$= \int_{-\infty}^{+\infty} \frac{x\epsilon^{-(x-a)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} dx \tag{65}$$

To solve this integral, a change of variables is required by letting:

$$y = \frac{(x-a)}{\sigma\sqrt{2}}\ ;\ dx = \sqrt{2}\sigma\ dy$$

then:

$$m_1 = \int_{-\infty}^{+\infty} \frac{(\sqrt{2}\,\sigma y + a)}{\sqrt{\pi}} \epsilon^{-y^2} dy \tag{66}$$

since

$$\int_{-\infty}^{+\infty} y\ \epsilon^{-y^2} dy \equiv 0 \tag{67}$$

and

$$\int_{-\infty}^{+\infty} \epsilon^{-y^2} dy \equiv \sqrt{\pi} \tag{68}$$

then:

$$m_1 = \frac{\sqrt{\pi}\ a}{\sqrt{\pi}} = a \tag{69}$$

In a similar manner, it can be shown that the second central moment or variance μ_2 is:

$$\mu_2 = \int_{-\infty}^{+\infty} \frac{(x - a)^2}{\sigma\sqrt{2\pi}} \epsilon^{-(x - a)^2/2\sigma^2} dx \tag{70}$$

Using the same change of limits as described in Reference 23, then:

$$\mu_2 = 2\int_{-\infty} \frac{\sqrt{2}\ \sigma^3 y^2}{\sigma\sqrt{2\pi}} \epsilon^{-y^2} dy \tag{71}$$

$$= \frac{4\sigma^2}{\sqrt{\pi}} \int_0^{\infty} y^2\ \epsilon^{-y^2} dy \tag{72}$$

$$= \frac{4\sigma^2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{4} = \sigma^2 \tag{73}$$

Likewise, it can be shown that the Gaussian distribution is properly normalized or that:

$$\int_{-\infty}^{+\infty} p(x)\ dx = 1 \tag{74}$$

thus:

$$\int_{-\infty}^{+\infty} \frac{\epsilon^{-(x\ -\ a)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\ dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \epsilon^{-y^2} dy \tag{75}$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \epsilon^{-y^2} dy = 1 \tag{76}$$

This Gaussian curve weighs values of x near a most heavily. The value of p(x) at the peak is $\frac{1}{\sigma\sqrt{2\pi}}$ so that as the width, σ, decreases, the height of the curve in the vicinity x = a, increases. Ultimately, as σ approaches zero, this curve approaches the delta function (x-a) and the variable x becomes a constant a, with a probability of 1. It is often desirable to know the relationship between the true rms value of a voltage which is normally distributed and the rectified average value of the same voltage. From the definition of the second moment, it has been shown that the true rms value of the Gaussian distribution is the standard deviation or simply σ. From the symmetry of the distribution, the average value can be found by

$$x_{av} = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{x\epsilon^{-x^2/2\sigma^2}}{\sigma}\ dx \tag{77}$$

Substituting $y = \frac{x}{\sqrt{2}\sigma}$ yields

$$x_{av} = 2\sqrt{\frac{2}{\pi}}\ \sigma \int_{0}^{\infty} y\ \epsilon^{-y^2}\ dy \tag{78}$$

$$= 2\sqrt{\frac{2}{\pi}}\left[\frac{\epsilon^{-y^2}}{(-2)}\right]_{0}^{\infty} \tag{79}$$

$$= \sqrt{\frac{2}{\pi}}\ \sigma = 0.798\ \sigma = x_{av} \tag{80}$$

Thus the ratio of rms to average (form factor) of a normally distributed

voltage is $\frac{\sigma}{0.798\sigma} = 1.25$.

It has already been shown that the cumulative probability distribution for a Gaussian density is given by:

$$p(x) = \int_{-\infty}^{x} \frac{\epsilon^{-(x - a)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} dx \tag{81}$$

and can be expressed in terms of the error function as:

$$p(x) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{x - a}{\sigma\sqrt{2}}\right)\right] \tag{82}$$

The average value of a Gaussian function cannot be defined in terms of a peak value since the peak values vary from zero to infinity. However, physical processes are usually only approximately Gaussian in that the peak values are limited to some value between three and ten times the rms value.

2.3.3.7 Rayleigh Distribution

It has been stated that a Rayleigh distribution arises in vibration work when the peak values of a narrowly filtered, normally distributed voltage are examined. This distribution is defined by:

$$p(x) = \frac{x\epsilon^{-x^2/2\sigma^2}}{\sigma^2} \tag{83}$$

Here, the random variable x refers to a peak value of the filtered noise signal and σ represents the rms value or standard deviation of the original unfiltered noise signal. Since the distribution is defined for positive x only, the mean value or first moment of the distribution is found to be nonzero and is

$$m_1 = \int_0^{\infty} x\ p(x)\ dx = \frac{1}{\sigma^2}\int_0^{\infty} x^2\ \epsilon^{-x^2/2\sigma^2}\ dx \tag{84}$$

Then, from entry 509 of Reference 23:

$$m_1 = \frac{1}{\sigma^2}\left[\frac{\sigma^2}{4}\right] \sqrt{2\pi\sigma^2} = \sqrt{\frac{\pi}{2}}\,\sigma \tag{85}$$

i.e., the average value of the envelope is 1.25 times the rms of the original noise wave. Similarly, the second moment can be evaluated as follows:

$$m_2 = \int_0^\infty x^2 \; p(x) \; dx \tag{86}$$

$$= \frac{1}{\sigma^2} \int_0^\infty x^3 \; \epsilon^{-x^2/2\sigma^2} \; dx \tag{87}$$

If we substitute $y = x^2/2\sigma^2$ then, from entry 508 of Reference 23:

$$m_2 = 2\sigma^2 \int_0^\infty y \; \epsilon^{-y} \; dy = 2\sigma^2 \tag{88}$$

That is, the mean square of the envelope is twice the mean square of the original noise wave. Also by direct substitution:

$$\sigma_R = \sqrt{m_2 - m_1^{\,2}} \tag{89}$$

$$\sigma_R = \sqrt{2\sigma^2 - \frac{\pi}{2}\sigma^2} = \sqrt{0.429\sigma^2} \tag{90}$$

Thus $\sigma_R = 0.655\sigma$ or the rms alternating current (a-c) component of the envelope is 0.655 of the rms of the original noise wave.

It can be shown that the Rayleigh distribution is properly normalized by evaluation of:

$$\int_0^\infty p(x) \; dx$$

This becomes:

$$\int_0^\infty \frac{x\epsilon^{-x^2/2\sigma^2}}{\sigma^2}\,dx = \int_0^\infty p(x)\,dx \tag{91}$$

which reduces to:

$$2\int_0^\infty y\ \epsilon^{-y^2}\,dy \tag{92}$$

then, from entry 429 of Reference 23:

$$2\int_0^\infty y\ \epsilon^{-y^2}\,dy \equiv 1 \tag{93}$$

The probability that the envelope exceeds some specified value x_o is obtained by integrating the density function from x_o to ∞. Thus:

$$\text{Prob. [envelope} > x_o] = \int_{x_o}^\infty p(x)\,dx \tag{94}$$

$$= \epsilon^{-x_o^2/2\sigma^2} \tag{95}$$

This is plotted in Figure 22.

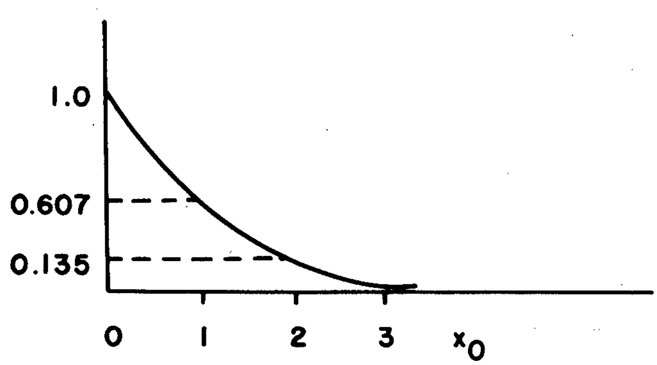


Figure 22. Probability That Envelope Exceeds x_o.
(Where x_o is Multiple of σ)

It is interesting to note that the Rayleigh distribution is independent of the original center frequency (about which narrow band filtering was performed) and is a function only of the mean square value σ^2 of the original unfiltered noise wave. Peak factors of the envelope density function have been partially tabulated. The tabulation is listed in Table 1.

Table 1. Density Function Peak Factors.

% Time Exceeded	Ratio	Ratio (db)
10	1.517	3.62
1	2.146	6.63
0.1	2.558	8.39
0.01	3.034	9.64
0.001	3.392	10.61
0.0001	3.675	11.31

Here the peak factor ratio is given as: $\frac{x}{\sigma_R}$

This table is for the peak factors of the Rayleigh distribution and should not be confused with the peak factor of the Gaussian distribution which applies to the original signal.

2.3.3.8 Determination of Distribution Functions

In practice, the characteristics of a signal x(t), at a specific time and at a specific frequency cannot be determined since only the instantaneous value of x(t) is known at a specific time. Instead, the characteristics of the signal in a given frequency bandwidth during a short period of time are determined. Thus, the first step in a statistical analysis is to select a time interval short enough so that the nature and magnitude of x(t) may be assumed constant, but long enough to give a statistically significant result. Often it is convenient to store the data sample on a continuous loop of magnetic tape, so that it may be played back repeatedly. Then a time sample of the signal may be analyzed to determine the probability distribution of either the instantaneous or the peak values of the signal. The concepts involved in this determination will be considered separately. The instantaneous values of the filtered signal are indicated by $x'(t)$ while the peak values of the filtered signal are indicated by $x'p(t)$. The instantaneous values and the peak values are described as follows:

a. Instantaneous Values - The probability distribution of instantaneous values is obtained for a set of discrete values of $x'(t)$

rather than as the continuous function that is illustrated in Figure 19. The probability distribution function at a value x'_1 is the probability $p(x' \geq)$ that x (t) exceeds x_1. (The filtered signal $x'(t)$ may be passed first through a mechanism for changing the polarity of the signal so that the positive and negative values of x(t) can be analyzed separately thus halving the number of discriminators and counters required for a given resolution. If this is done, the time-history must be played back for twice as long but this usually causes only a small increase in the total analysis time.) Refer to Figure 23 and note that this is equivalent to summing all the $\Delta t'$ values during which x (t) exceeds x_1, or level L_1, and dividing by the total elapsed time. If this is done for a number of selected levels, L_1, L_2, ... L_n, the probability distribution function at these levels is obtained. Figure 24 illustrates the block diagram of a statistical analysis measurements system. The filtered signal is fed to an array of discriminators which determine whether a signal is greater or less than a preselected value. The levels of the discriminators are set at L_1, L_2, L_3, etc. usually in equal increments of voltage. The detail or resolution obtained is governed by the number of discriminator levels selected for measurement. The detail and accuracy yielded by ten levels for each polarity are sufficient for many engineering purposes. The resolution obtained improves with the number of counters (shown in Figure 24) registering. Therefore, a variable gain control is needed to adjust the signal level, by a known amount, until an adequate number of counters are registering. When the signal $x'(t)$ exceeds the level of a particular discriminator, a clock is started. It is stopped when $x'(t)$ next falls below that level, and the counter counts the elapsed time between these events. At the end of the data sample, the readings of the counters are recorded by the read-out mechanism such as a digital counter. If each reading is divided by the total elapsed time of the data sample, the values obtained are the values of the probability distribution function at the values of the levels L_0, L_1, L_2, etc. If the values at adjacent levels are subtracted, the average probability density function between these two levels is obtained. The polarity of the signal is now changed and the process is repeated. However, using positive polarity, the distribution function, $p(x' \geq)$ is obtained, while using the negative polarity yields $p(x' \leq)$. Since $p(x' \leq) = 1 - p(x' \geq)$, the two sets of values may be combined to give the complete distribution function. The character of the signal can then be evaluated by the use of probability graph paper.

In addition to determining the characteristics of the signal, this analysis can be used to determine the mean square values of $x'(t)$, either by use of the probability paper, if the signal is Gaussian, or in any case, by computing the moment of inertia (second moment) of the probability density function about the mean value as follows:

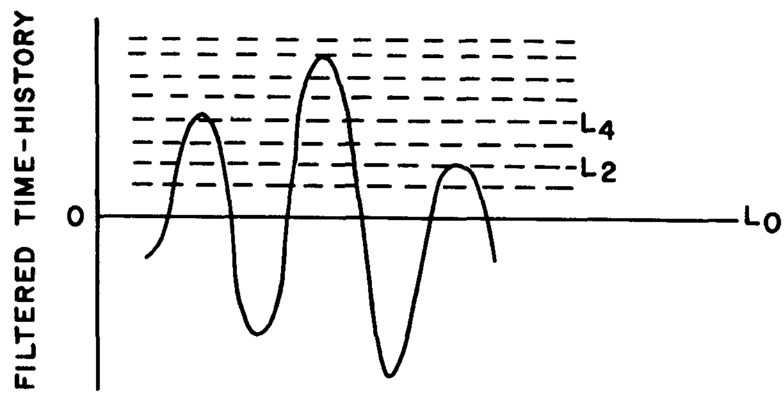


Figure 23. Measurements Required for Statistical Analysis.

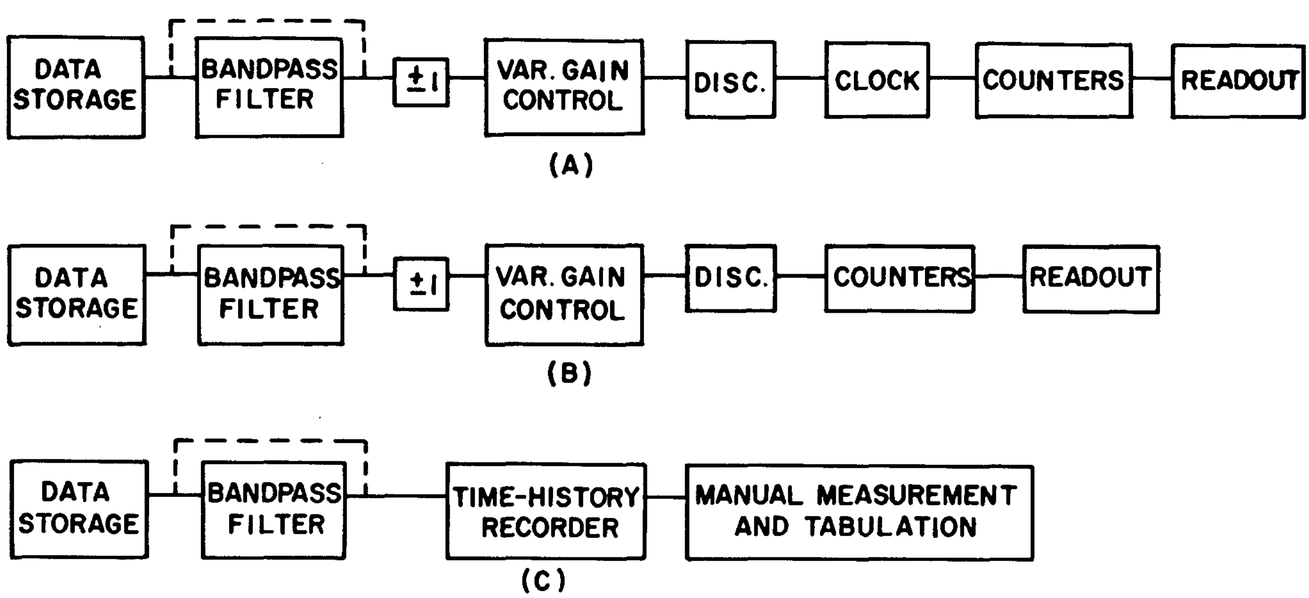


Figure 24. Block Diagrams Showing Steps in Statistical Analysis.

$$\sigma^2 = \int_{-\infty}^{+\infty} (x - x)^2 \; p(\bar{x}) \; dx \tag{96}$$

$$= m_2 - m_1{}^2 \tag{97}$$

b. Peak Values - The probability distribution of peak values or maxima $x'p(t)$ of the filtered signal $x'(t)$ is obtained for a set of discrete values of $x'p(t)$ rather than as the continuous function illustrated in Figure 19. The probability distribution function at a value $x'p_1$, is the probability $p(x'p_1 \geq)$ that a peak or maximum of $x'(t)$ exceeds $x'p_1$. Refer to Figure 23 and note that this is equivalent to counting the number of times that $x'(t)$ exceeds $x'p_1$, (i.e., level L_1) and dividing by the total number of times that $x'(t)$ exceeds L_0. If this count is made for a number of levels L_1, L_2 ... L_n, the probability distribution function at these values is obtained.

A block diagram of a system used for obtaining signal peak measurements is shown in B of Figure 24. The filtered signal $x'(t)$ passes through a bank of level discriminators and causes the associated counters to register one count each time the level of the particular discriminator is exceeded. Sometimes an array of discriminator-counter combinations is called a "pulse-height analyzer"). At the end of the data sample, the readings of the counters are recorded. Division of each count by the zero level count yields the probability distribution function $p(x'p \geq)$ at the values equivalent to the levels L_1, L_2, ... L_n. If the values at adjacent levels are subtracted, the average probability density function between these two levels is obtained. The character of the signal can then be evaluated by the use of probability graph paper.

For some end uses of the data, such as fatigue analysis, the probability distribution of the total or unfiltered signal $x(t)$ rather than the probability distribution of the filtered signal $x'(t)$ is desired. Then, the distribution of the peaks of $x(t)$ is obtained by shunting out the filter as indicated by the dotted lines in Figure 24.

The systems shown in A and B of Figure 24 are relatively complex, and unless a large amount of analyses of this type are to be made, the cost of such systems may not be justified. A less complex method is shown in C of Figure 24. In this case, a direct writing recorder is employed and the number of peaks which exceed certain levels (such as shown in Figure 23) are counted manually. In principle, the distribution of instantaneous values may be measured in this way but practical difficulties make this method difficult to employ.

Recently, techniques have been developed which permit estimation of the probability density or distribution function of broadband random signals with accuracies of better than one percent. Signals in the frequency range of d-c to 10 kc are analyzed by means of a slicer, or window, circuit which passes pulses of a radio frequency (RF) oscillator output, normally 10 megacycles (mc) every time the signal under study passes through the specified window. By automatically sweeping the window levels and counting the cycles of RF passed through the selecting circuit as compared to the total number of cycles of RF passed through the selecting circuit as compared to the total number of cycles of RF generated, accurate, automatic displays of probability density within the range of ± 5 sigma have been obtained.

2.3.4 Frequency Analysis

The objective of spectral analysis is to determine the variation of vibration magnitude with frequency. The narrowness of the bandwidth of the filter employed in the analysis determines the frequency resolution of the analysis and, therefore, the ability to detect the "fine-grain" variation of magnitude with frequency. The magnitude obtained at a particular frequency will be the average magnitude over the short time interval of data analyzed during which the nature and magnitude of the vibration may be considered constant. (See Figures 2 and 4.)

In preparing for spectral analysis, the characteristics of the time-history x(t) should be known either from statistical analysis or from previous experience so that the most appropriate units can be selected for describing the vibration magnitude. In practice, the spectral analysis often is conducted first and the necessity for statistical analysis is determined from the resulting spectrum.

2.3.4.1 What Value to Plot

To achieve the greatest flexibility from vibration data and make it amenable to comparison with data analyzed on other systems, it should be plotted on a per-cycle basis. In other words, the quantities used to express the severity of vibration in different parts of the spectrum should be divided by the effective filter bandwidth. Further, any single number used to describe the vibration within a given frequency band, or in the entire spectrum, must be obtainable by an rms operation, from a plot of the spectrum.

In the determination, of which single number quantity should be used to express the vibration in a specified band, three considerations favor the rms of the instantaneous accelerations.

a. The rms value by definition is the standard deviation of the instantaneous accelerations about zero as a mean and lends itself conveniently to statistical analysis.

b. The rms values for two or more random collections, even though they are in different frequency bands, generally may be combined as the square root of the sum of the squares.

c. The rms value is the widely accepted standard measure of a-c voltage in electrical circuitry.

Plotting the function, W(f) against frequency, it should be possible to calculate the rms acceleration, σ, between any two frequencies, f_1 and f_2,

by a simple operation, based on the requirement that the rms value for an entire frequency band be the square root of the sum of the mean square values for all bands within. This is satisfied if W(f) is defined as:

$$W(f) = \lim_{\Delta f \to 0} \frac{\sigma \Delta f^2}{\Delta f} \tag{98}$$

where $\sigma\Delta f^2$ is the mean square acceleration in the rectangular bandwidths Δf.

It follows then that the function W(f), called the mean square acceleration density, in g^2 per cycles per second (cps), is the most convenient means of describing the frequency distribution of a continuous spectrum.

A value which can be used to plot frequency distribution of power is "power spectral density". Power, which is the rate of doing work, is proportional to the square of the amplitude of a harmonic vibration. If two frequencies are present in a vibration, the power is proportional to the sum of the squares of the individual amplitudes associated with the two frequencies.

A random vibration can be considered to be a sum of a large number (tending to infinity) of harmonic vibrations of appropriate amplitudes and phase. The total power is again the sum of the component harmonic vibrations. The way in which this power is distributed as a function of frequency, is known as "power spectral density" and it represents the power per unit frequency interval. A plot of this quantity indicates the frequency distribution of power.

2.3.4.2 Definition of Power Spectral Density

Power spectral density is defined as the limiting value of the mean square response $[\overline{x'}]^2$ of an ideal bandpass filter to x(t), divided by the bandwidth B of the filter as the bandwidth of the filter approaches zero. (An ideal bandpass filter has a transmission characteristic which is rectangular in shape so that all frequency components within the filter bandwidth are passed with unity gain and zero phase distortion, while frequency components outside the bandwidth are completely removed. If the transmission characteristic is H instead of unity in the bandwidth B, the spectral density is obtained by dividing the mean square response by B x H^2 instead of B.)

An alternative definition is as follows: If the function x(t) is passed through an ideal low pass filter with cutoff frequency f_c, the mean square response of the filter $[\overline{x'}]^2$ will increase or decrease as f_c is increased or decreased, i.e., more or less of the function will be passed by the filter (assuming f_c is varied in a frequency range where the power spectral density is nonzero). (An ideal low pass filter is an ideal bandpass filter, having a lower cutoff frequency of zero.)

The power spectral density W(f) is the rate of change of $[\overline{x'}]^2$ with respect to f_c, i.e.,

$$W(f_c) = \frac{d}{df_c}\left[\overline{(x')^2}\right] \qquad (99)$$

Figure 25 illustrates a plot of power spectral density as a function of frequency, obtained, for example, from spectral analysis of a random function.

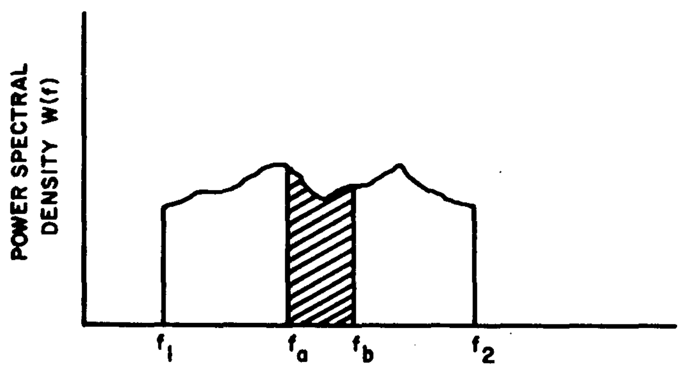


Figure 25. Typical Power Spectral Density Plot of Broadband Random Function x(t).

The mean square value, or variance, of the frequency content of x(t) between the frequencies f_a and f_b is:

$$\sigma^2\ (f_a \le f \le f_b) = \int_{f_a}^{f_b} W(f)\, df \qquad (100)$$

The rms value is σ. The mean square value of the complete function $[\overline{x}]^2$ is given by equation (100), when f_a and f_b are zero and infinity respectively; it is equal to the area under the entire spectrum. In the case of white noise, for which the spectral density is independent of frequency, i.e., W(f) = W, equation (100) simplifies to:

$$\sigma^2 = W\ (f_2 - f_1) \qquad (101)$$

where f_2 and f_1 are the limiting frequencies of the noise.

To understand the physical meaning of the power density spectrum, consider one simple method by which it may be theoretically measured. Suppose that the power density spectrum of the random voltage wave $f_1(t)$ shown in Figure 1 is required. For simplicity, assume that the random wave contains no periodic or d-c components. To determine the spectrum, the random wave is applied to an ideal, conventional type, low pass filter with a 1-ohm resistor output. The filter cutoff frequency is allowed to be continuously adjustable over the important range of frequencies of the random wave. Since the filter is ideal, all components of the random wave below the cutoff frequency pass the network attenuated and all others are completely eliminated by the network. By means of an appropriate power measuring device, the power consumed by the resistor is determined at various cutoff angular frequencies ω_1, ω_2, ω_3 ... and plotted as shown in Figure 26. Clearly, a point on this curve at a given angular frequency ω indicates power consumption in the frequency range between zero and the given frequency. If the slope of this curve is determined graphically as a function of ω, the result would appear as the dotted curve shown in Figure 27. This curve of slope versus angular frequency is a spectrum of power density because its integral from zero to a given value of ω is the power due to the aggregate of components within the range of frequencies under consideration.

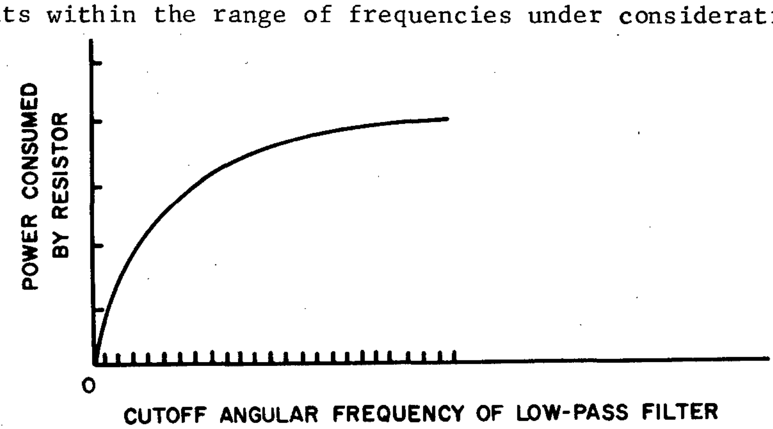


Figure 26. Curve of Power Consumed by Resistor Versus Cutoff Angular Frequency of Ideal Low-Pass Filter.

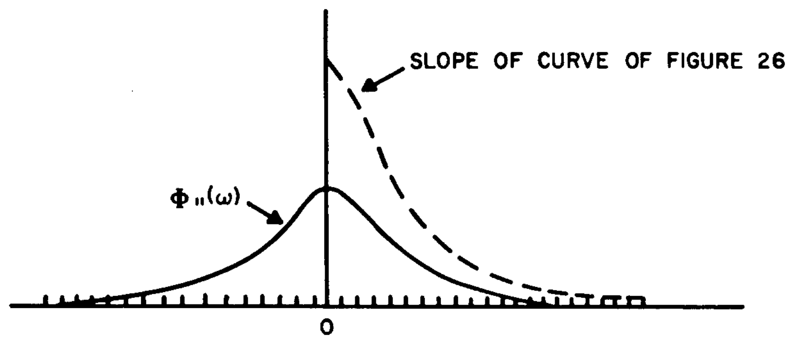


Figure 27. Slope of Curve of Figure 26 and Power Density Spectrum of Random Wave Figure 1.

In other words, the measured curve of Figure 26 is an integrated power spectrum so that its derivative as given in Figure 27 is a power density spectrum. Finally, in order that the power density spectrum thus obtained be in agreement with its theoretical definition, the amplitude of the curve for positive frequencies is halved and the upper half is drawn for negative frequencies of corresponding values. The result is the curve $\Phi_{11}(\omega)$ of Figure 27 which is an even function as it should be. Negative frequencies, although physically nonexistent, have been introduced in harmonic analysis for the reason that they make possible the expression of Fourier series and integrals in exponential forms which are simple and compact. It is important that expressions involving negative frequencies in a physical problem be correctly interpreted. Note on Figure 27, that where ω equals 0, the curve of $\Phi_{11}(\omega)$ has a finite value; however, this does not mean that the random voltage has a finite d-c component. The reason is that $\Phi_{11}(\omega)$ equals $[W(\omega)]$ is a spectrum of power density so that the d-c power according to equation (116) should be

$$W(o)\ d\omega$$

which is an infinitesimal quantity. Actually, this amount includes the contributions of all components in the band $d\omega$ in the neighborhood where ω equals o. Since the area under the curve of $\Phi_{11}(\omega)$, not its amplitude, represents power, the amount of power in any band of frequencies finite or infinite, is finite, but the power in an infinitestimal band ω is infinitesimal. Furthermore, the power at a particular frequency is zero when the spectrum is a continuous density spectrum.

2.3.4.3 Relationship of Power Spectral Density to Line Spectrum

The mathematical relationship between power spectral density W(f) and the Fourier coefficients C_n is:

$$W(f) = 2\pi \lim_{B \to o} \frac{\overline{|x'|^2}}{B} = 2\pi \frac{d}{d\omega} [\overline{(x')^2}] \tag{102}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} Cn^2 \ \delta\ (f - fn) \qquad [b_o = 1] \tag{103}$$

where $[\overline{x'}]^2$ is the mean square response of the ideal filter with bandwidth B and center frequency f, where f equals $\frac{\omega}{2\pi}$; δ (f - fn) is the Dirac delta function.

The power spectral density of a sinusoidal function is a line spectrum since such a function has effectively zero bandwidth. $[\overline{x'}]^2$ will change instantaneously as the cutoff frequency f_c increases through the frequency of the sine wave giving theoretically an infinite value for W(f).

2.3.4.4 Continuous Spectrum Versus Line Spectrum

Full information concerning harmonic amplitudes and phases is contained in the function F(n), which in general is complex and is called the complex spectrum of f(t). Therefore:

$$f(t) = \sum_{n=-\infty}^{+\infty} F(n) \epsilon^{jn\omega_1 t} \tag{104}$$

$$F(n) = \frac{1}{T_1} = \int_{\frac{-T}{2}}^{\frac{T}{2}} f(t) \epsilon^{-jn\omega_1 t}\, dt \tag{105}$$

Because the harmonic order n, assumes only discrete values, the spectrum is a line spectrum.

Inasmuch as equation (104) is a summation of sinusoids in accordance with the amplitude and phase information contained in the complex spectrum, F(n), for regaining the original function, it is a synthesis in opposition to equation (105) which is an analysis. Again, whereas equation (105) is a transformation of f(t) into its frequency-domain representation F(n), equation (104) is an inverse transformation of F(n) into its time-domain representation f(t) and is called the Fourier transform of F(n).

Relationships (104) and (105) for a periodic function are, therefore, Fourier transforms of each other and from equation (105) we get a complex line spectrum. If equations (104) and (105) are combined, the result is:

$$f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \epsilon^{jn\omega_1 t} \; \omega_1 \int_{-\frac{T}{2}}^{\frac{T}{2}} f(\sigma) \; \epsilon^{-jn\omega_1 \sigma} d\sigma \qquad (106)$$

where σ is a dummy variable introduced for the sake of nonambiguity. Now, if the period T_1 grows without limit, the periodic function f(t) tends to an aperiodic one and in so doing, equation (106) approaches a limiting form. As T_1 tends to infinity, the fundamental angular frequency ω_1 becomes a differential of angular frequency $d\omega$ and $n\omega_1$ which is the nth harmonic angular frequency, becomes the continuous angular frequency ω_1 and the summation over all harmonics becomes an integration over the entire continuous frequency range $(-\infty, \infty)$. Therefore, the limiting form for equation (106) for an aperiodic function f(t) is:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \epsilon^{j\omega t} \, d\omega \int_{-\infty}^{+\infty} f(\sigma) \epsilon^{-j\omega\sigma} \, d\sigma \qquad (107)$$

This is the Fourier integral for an aperiodic function. Writing two reciprocals from equation (107) as was done for the case of Fourier series yields:

$$f(t) = \int_{-\infty}^{+\infty} F(\omega) \epsilon^{j\omega t} \, d\omega \qquad (108)$$

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) \epsilon^{-j\omega t} \, dt \qquad (109)$$

Here $F(\omega)$ is a continuous function of the angular frequency ω and is in general complex. It is the complex continuous spectrum of the aperiodic function and the relations (108) and (109) are known as Fourier transforms.

2.3.5 Determination of Power Spectral Density

The analysis of random vibration always involves a record of finite length. When a sample of time-history of finite duration is employed to compute the power spectral density of a random function, it is assumed that:

a. The function is ergodic, i.e., that averaging one time-history with respect to time yields the same result as averaging over an ensemble of time-histories at a given instant of time.

b. That the function is stationary, i.e., that the power spectral density is independent of the sample of time-history chosen.

Further, the averaging time or sample duration must be long enough to yield a statistically significant value. Thus, the mean square obtained should not vary appreciably with a change in averaging time. The time over which a vibration record may be considered a stationary process and the need for a sufficiently long averaging time often are conflicting requirements.

Assume that a magnetic tape record of finite length is to be analyzed by an electronic analyzer as a stationary process. The tape can then be formed into a loop of arbitrary length and made to repeat indefinitely on the analyzer. By such a procedure, an arbitrary fundamental period corresponding to the length of the loop is established and the contents of the record may be defined in terms of the Fourier coefficients of integral multiple harmonics of the loop frequency ω_o.

The function x(t) then can be represented by the real part of the Fourier series as:

$$x(t) = \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} C_n \epsilon^{jn\omega_o t}\right]$$

$$= C_o + 2\sum_{n=1}^{\infty} \left|C_n\right| \cos\,(n\omega_o t - \alpha n) \tag{110}$$

where the complex amplitude C_n is defined by the equation:

$$C_n = \frac{1}{2T} = \int_{-T}^{T} x(y)\epsilon^{-jn\omega_o y}\,dy \tag{111}$$

and 2T is the loop period.

The mean square value $\overline{x^2}$ is a stationary property of x(t) that

is determined by equation (6) as:

$$\overline{x^2} = \frac{1}{2T}\int_{-T}^{T} [x(t)]^2 \, dt$$

$$= C_o^{\,2} + 2\sum_{n=1}^{\infty} |C_n|^2 \qquad (112)$$

This equation indicates that the contribution to the mean square value in any frequency interval is the summation of all the components within that frequency band.

If now the length of the loop and the loop period are doubled, the number of spectral lines will be doubled. Since the mean square value of the entire spectrum must remain essentially constant, the values of the new coefficients $|C'_n|^2$ must decrease to approximately one half their former values as illustrated in Figure 28.

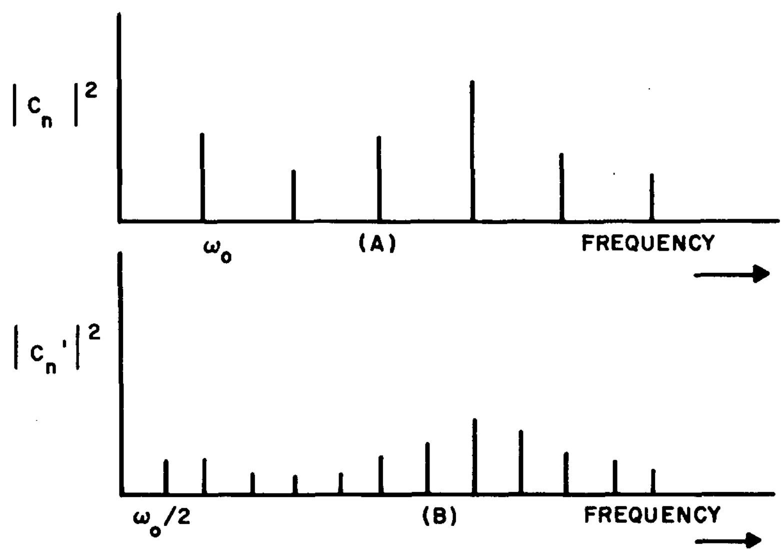


Figure 28. Spectrum Analysis of Random Vibration, As Determined from a Magnetic Tape Loop.

The sum of the lengths of the $|C_n|^2$ lines up to any frequency is the same in each diagram of Figure 28, and the increment in x^2 divided by the increment in frequency is essentially the same in each case. Thus:

$$\frac{\overline{\Delta x^2}}{\Delta\omega} = \frac{2|C_n|^2}{\omega_o} = \frac{2|C'_n|^2}{\frac{1}{2}\omega_o} = W(\omega_n) \tag{113}$$

The quantity $W(\omega_n)$ is the power spectral density at $\omega = \bar{\omega}_n$. The dimensions are the square of the parameter represented by $\overline{\Delta x^2}$ in equation (113) per unit of frequency.

As the length of the loop is increased, more and more spectral lines are introduced; the magnitude of $|C_n|^2$ decreases correspondingly but $W(\omega_n)$ remains finite. In the limiting case, as T approaches infinity or $\Delta\omega$ approaches zero, the spectrum becomes continuous and the discrete values of $W(\omega_n)$ approach a smooth function $W(\omega)$:

$$W(\omega) = \lim_{\Delta\omega\to o} \frac{\overline{\Delta x^2}}{\Delta\omega} = \frac{\overline{dx^2}}{d\omega} \tag{114}$$

The mean square value can be expressed by the integral:

$$\overline{x^2} = \int_o^\infty W(\omega)\, d\omega \tag{115}$$

where only positive frequencies are considered. This is equivalent to the definition over the entire range of frequencies given by

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [f_1(t)]^2\, dt = \int_{-\infty}^{+\infty} W(\omega)\, d\omega \tag{116}$$

The power spectral density of a random time function may be determined by feeding the function x(t) into a spectrum analyzer that transmits only those frequency components within the passband of the analyzer,

$\omega \pm \frac{\Delta\omega}{2}$ as shown in Figure 29. The output in the passband, namely, $\overline{\Delta x^2}$ is then indicated by a "mean square" meter. The mean power spectral density $W(\bar{\omega})$ over the passband at $\bar{\omega}$ is determined by dividing the meter reading by $\Delta\omega$. This division may be incorporated directly in the calibration of the meter. The complete distribution of $W(\omega)$ then is determined by changing the frequency $\bar{\omega}$ in increments of $\Delta\omega$. The mean square and averaging methods often used to determine power spectral density are shown in Figures 32 and 33.

2.3.5.1 Measurement of Power Spectral Density by the Use of Filters

When a random function is applied to the input of a filter, the output is not, in general, represented by the normally defined 3-decibel (db) bandwidth. Rather, the "noise bandwidth" of the filter is involved and the output is obtained by multiplying the input spectral density by the square of the transmission characteristic of the filter. An ideal bandpass filter has a bandwidth B and an output spectral density of W times B if W is the constant spectral density input. In a typical case however, the effective or noise bandwidth of a filter should be determined by the mean square response of the filter to known white noise. If a random signal is applied to an ideal filter, the power spectral density output is the mean square output divided by B. In a practical case, the mean square output divided by the effective filter bandwidth is a good approximation of the average spectral density within the filter bandwidth.

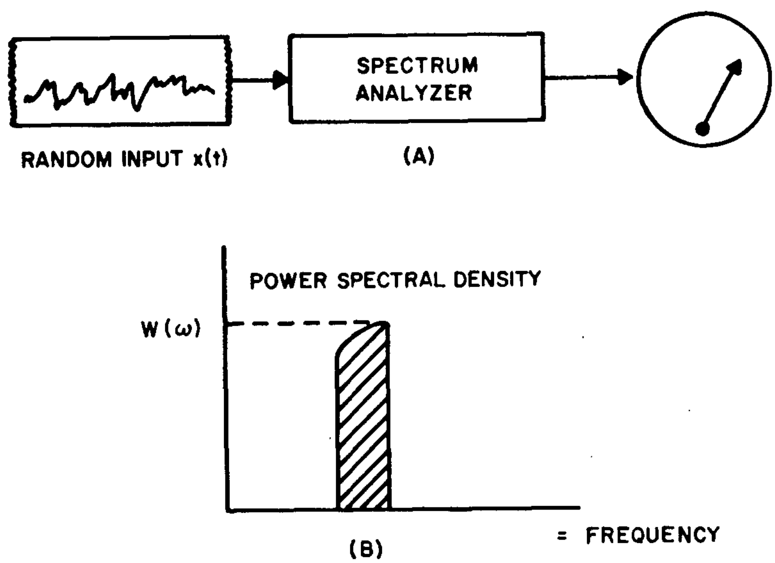


Figure 29. The Experimental Determination of The Power Spectral Density $W(\omega)$ of a Random Function $x(t)$.

2.3.5.2 Effects of Filter Characteristics

The properties of filters used for the measurement of power spectral density, such as transmission pattern characteristics, time constant, and the rate of variation of spectral density with frequency of the signal under study will have a most significant effect on the resulting spectrum plot. If the filter bandwidth is wider than the most narrow resonance under study, then although analysis time might be reduced, there is insufficient resolution for an accurate analysis. In general, the effect of too wide a filter bandwith is to smooth the resulting plot. The output may in some cases, be time averaged by the effect of the filter time constant. The effect of too wide a filter (dotted lines) and how it can mask peaks and notches is shown in Figure 30.

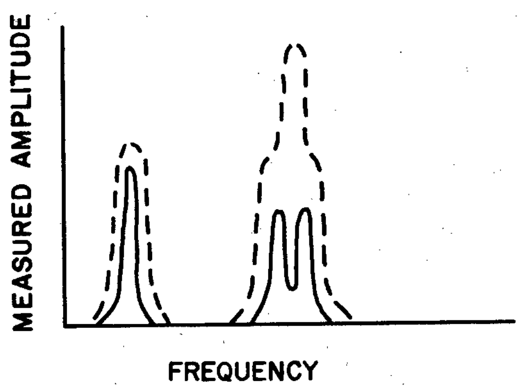


Figure 30. Effects of Filter Bandwidth Characteristics on Measured Spectra.

2.3.5.3 Measurement of Line Spectra by the Use of Filters

Sinusoidal functions are analyzed with filters by plotting filter output amplitudes instead of spectral density. The accuracy of the measurement depends critically on the filter bandwidth, and on the readout device. Two sinusoids narrowly separated in the spectrum are detectable only with sufficiently narrow filtering devices (high selectivity). For a random wave applied to a narrow band filter, the output is often considered to be the mean square value of a single sinusoid rather than the mean square value of the sum of many sinusoids.

2.3.5.4 Spectrum of Sinusoid Plus Noise

If the random wave under observation has a periodic component, then

either the power spectrum or power density spectrum could prove inadequate. Strictly speaking, the power density spectrum, although well suited for characterizing a random wave, does not give a convenient representation of a periodic component because of the infinite discontinuities involved at the "line spectrum" points of the periodicity. Similarly, the random component is not easily represented by the power spectrum. For example, in Figure 31, the peak could be caused either by a periodicity or a truly random function of increased intensity over that frequency band. In such cases, a periodicity, if present, can be easily detected in the probability density function of the signal under study. Presence of a periodicity can also be detected in the integrated power spectrum, defined as:

$$\int_{-\infty}^{\omega} W(\omega)\, d\omega \tag{117}$$

This function increases monotonically for a random wave but has finite discontinuities at the harmonic frequencies of a periodic wave.

2.3.5.4 Measurement of Filter Output

The magnitude of the filter output can be obtained in terms of the mean square or average value of the filtered signal. If statistical analysis has shown $x'(t)$, the filtered signal, to be a random function then the mean square value is determined by squaring and integrating the function and then dividing by integration time T as shown in Figure 32. Division of the output of the integrator or alternate averaging device (shown dotted) by the effective filter bandwidth yields the average power spectral density within the filter bandwidth. When the filter output is simply rectified and smoothed, the average value of $x'(t)$ is obtained. If this is known to be a random function, power spectral density can be obtained by the relationships described in paragraph 2.3.3 or by using the empirical measurement of response to a known spectral density. Spectral analysis is accomplished by the averaging method shown in Figure 33.

2.3.6 Magnitude Analysis

2.3.6.1 Magnitude Time-History Analysis

In this type of analysis, the variation of the vibration severity as a function of time is examined. The result may appear as in Figure 34. Often the magnitude is computed as the average over a few seconds while the data record itself may last for several minutes. Either the magnitude of the complete, unfiltered signal may be examined, such as the overall mean square value $\overline{x^2}$, or the magnitude of the vibration in a restricted band of frequencies can be determined. If a spectrum analysis shows that the important components of a given vibration signal exist only over selective frequency intervals, then the magnitude time-history analysis would be carried out only for those

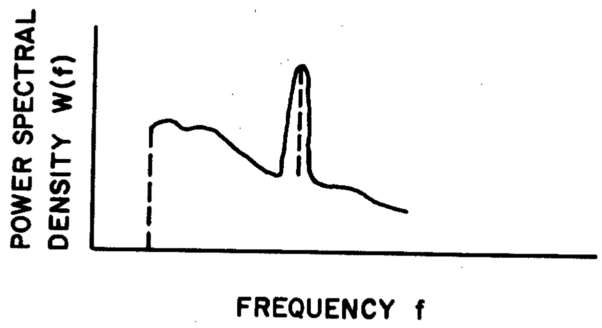


Figure 31. Example of Line Spectrum Superimposed on Random Vibration Spectrum.

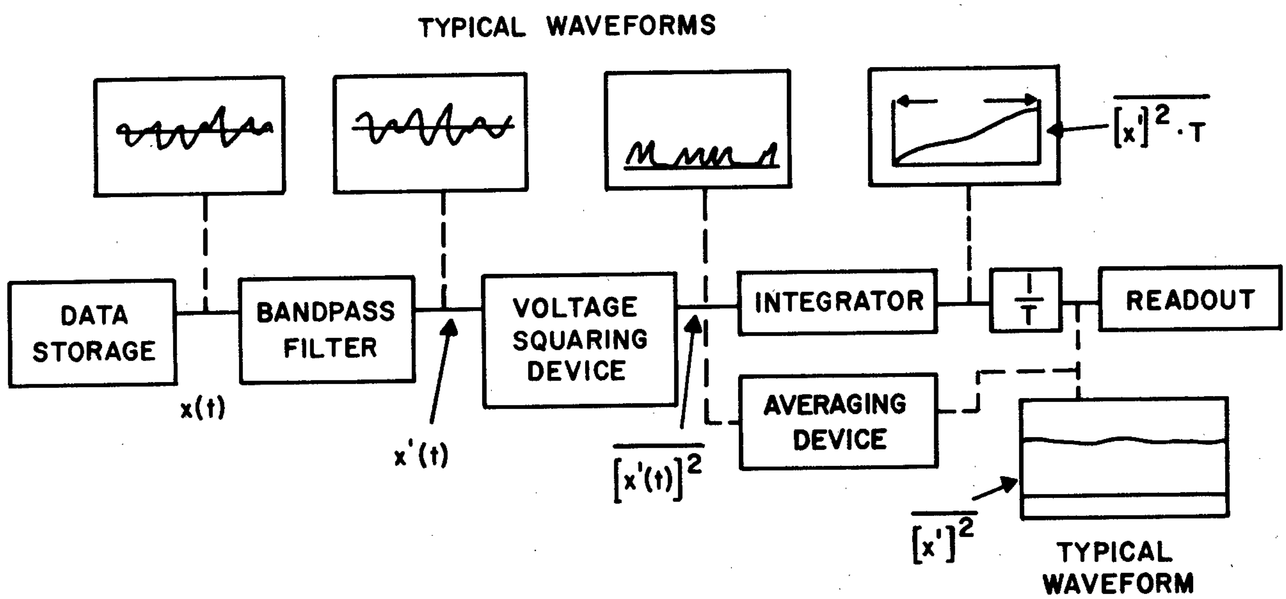


Figure 32. Block Diagram Showing Spectral Analysis by the Mean-Square Method.

frequencies. For example, the amount of analysis necessary to define the surface of Figure 2 depends on the degree of the completeness with which the surface is defined.

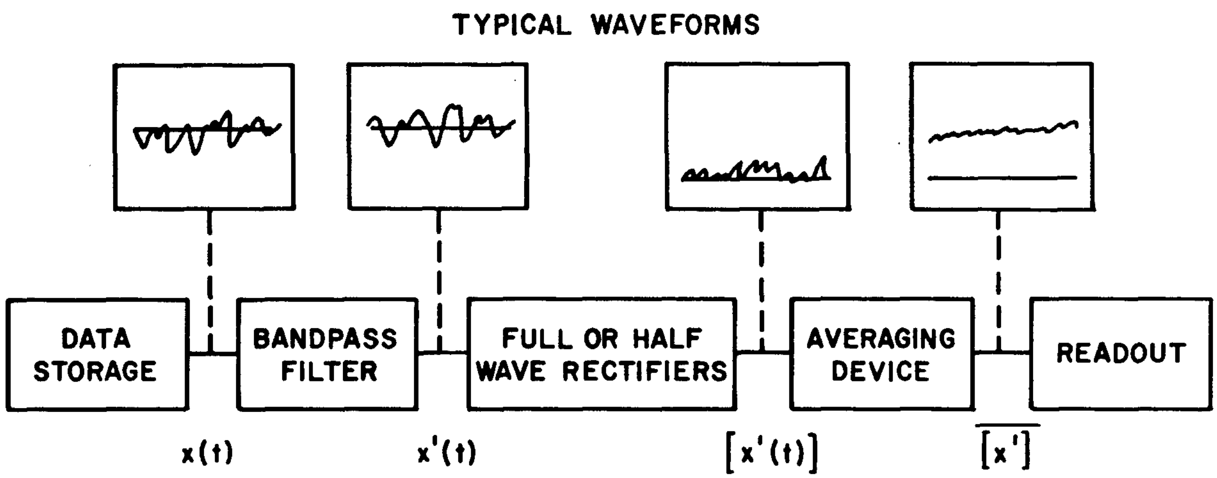


Figure 33. Block Diagram Showing Spectral Analysis by Averaging Method.

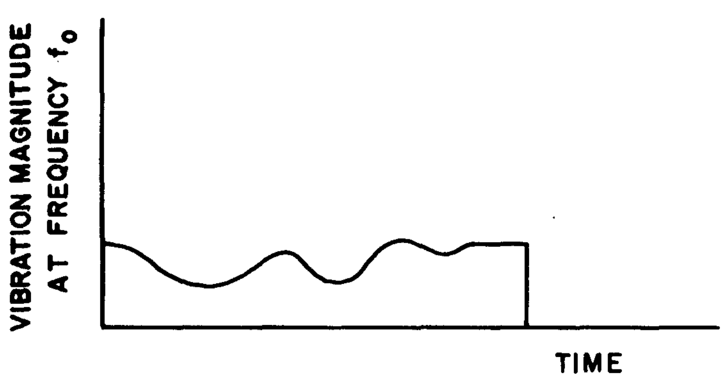


Figure 34. Typical Plot of Magnitude Time-History Analysis.

2.3.6.2 Magnitude Distribution Function

The magnitude distribution function of a signal is the total time that

the magnitude is equal to or greater than a particular magnitude. The mean square or mean value of the signal is measured on a bank of discriminator-counter or discriminator-clock-counter combinations, dependent on the averaging used. (See Figure 35.) Figure 36 illustrates one method of plotting the readings from the counters. The ordinate corresponds to the magnitude of the level which is exceeded whereas the abscissa is normalized to indicate the percentage of total time that the level is exceeded. The function illustrated in Figure 36 is analogous to probability density as shown in Figure 19. Unless the analysis represents the total, unfiltered signal, the filter center frequency and bandwidth must be specified. The relations between times of occurrences of different vibration magnitudes is lost with this type of presentation, similar to the way time is averaged out of spectrum presentation on all but real time analyzers as described in Reference 30.

2.3.7 Correlation Analysis

A correlation function describes the time relation between two parameters as a function of the time delay between the parameters at the times in which they are observed. Physically, correlation involves the delay of one signal with respect to another, multiplication of the two signals and integration of the product over suitable limits. If the two signals represent the same phenomena, but delayed in one case with respect to itself, then the process is called autocorrelation. For example, $x_1(t)$ and $x_1(t + \tau)$, where $x_1(t)$ is a physical event such as missile vibration at a specific point. If the two functions are physically distinct from one another (for example, $x_1(t)$ and $x_2(t + \tau)$ then the process of delay, multiplication, and summation is known as crosscorrelation.

Correlation analysis can be applied to missile vibrations in several ways. For example, the propagation of vibration through a structure can be studied by examining the crosscorrelation function between acceleration measurements at two different points of the structure. Crosscorrelation between parameters with different physical units is also possible; as for example the crosscorrelation between applied force and resulting acceleration at points of a missile.

2.3.7.1 Autocorrelation Function

The autocorrelation function of a signal x(t) also known as the average lag product $R(\tau)$ is defined as:

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(\tau)\, x(t + \tau)\, dt \tag{118}$$

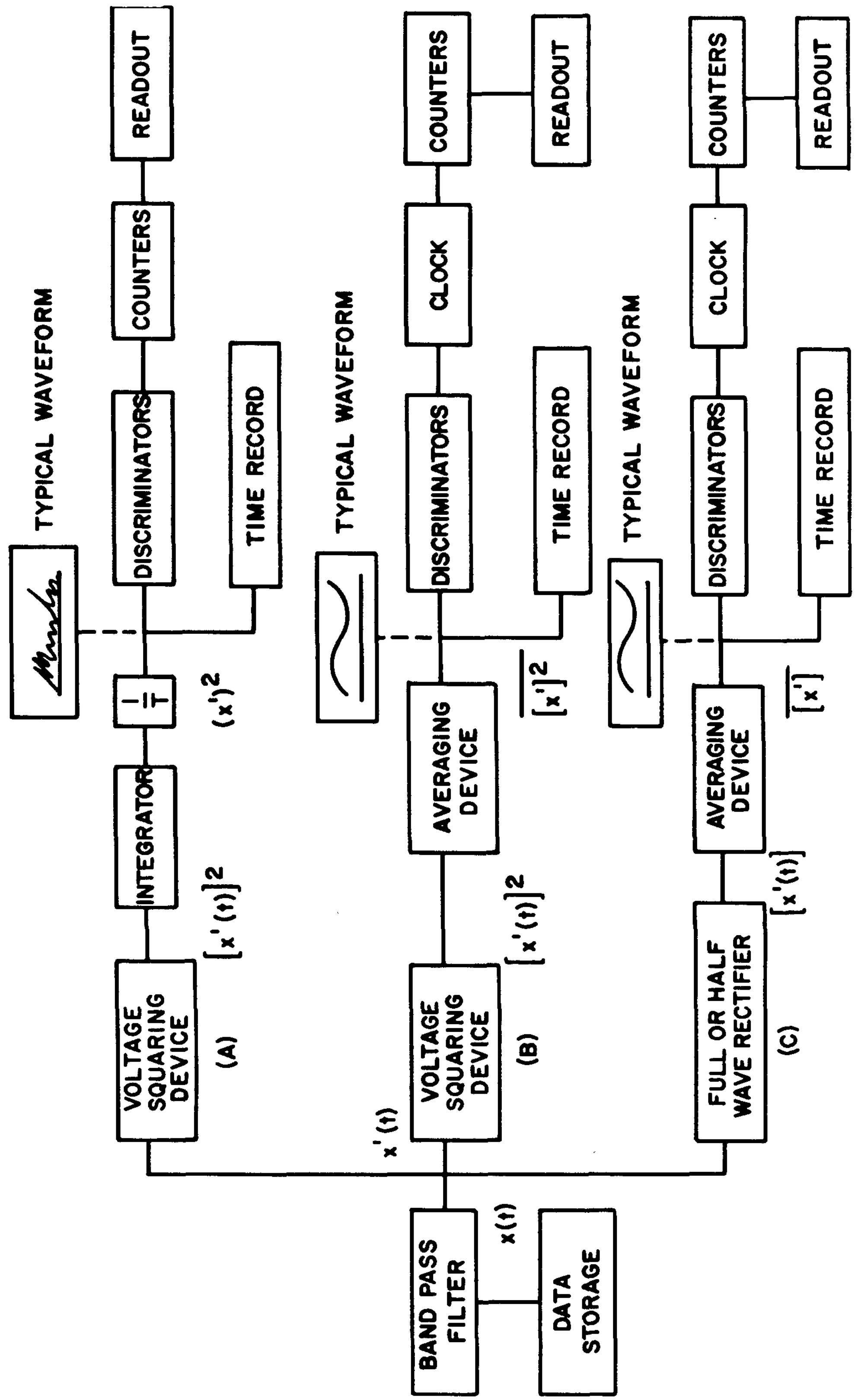


Figure 35. Block Diagram Showing Magnitude Time-History Analysis.

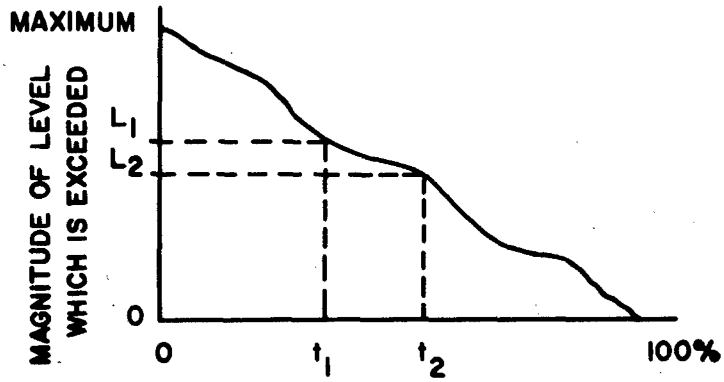


Figure 36. Distribution Function of Vibration Magnitude for a Filter Having a Center Frequency f_o

and is seen to be a function of the time delay τ. For strictly statistical usage, division by the normalizing factor R(o) is necessary, so that the value of the function is unity at zero lag. Under this condition, it can be called the autocorrelation function. However, widespread use has brought about the name autocorrelation function for $R(\tau)$ and not the ratio $R(\tau)/R(o)$.

Several useful properties are associated with the autocorrelation function:

a. $R(\tau) = R(-\tau)$ i.e., it is always an even function.

b. $R(o) = \sigma^2 + m_1{}^2$ i.e., it equals the average power in a sample function or the sum of a-c and d-c powers.

c. $R(o) \geq R(\tau)$ i.e., the largest value of the autocorrelation function occurs at zero lag since a function is never more closely related to another function than it is to itself.

d. $|R(\tau)| \leq 1$ i.e., the autocorrelation function can never exceed plus or minus one in magnitude.

e. The Fourier transform of the autocorrelation function is the power density spectrum. In other words, if W(f) is the one-sided power spectral density then:

$$R(\tau) = \frac{1}{2\pi} \int_0^\infty W(f) \cos \omega\tau \, d\omega \qquad (119)$$

and

$$W(f) = 4 \int_0^\infty R(\tau) \cos \omega\tau \, d\tau \qquad (120)$$

The last two equations are known as the Wiener-Khintchine relations and form a Fourier transform pair.

Equation (120) can be used to determine the power spectral density of a random function from the autocorrelation function.

f. $R(\infty) = 0$, if $x(t)$ contains no periodic or d-c components.

g. If the autocorrelation function is continuous at the origin, it is continuous everywhere.

One of the most important characteristics of the autocorrelation function is that even though retaining all the harmonics of the given function and containing no new ones, it discards all their phase angles. In other words, all periodic functions having the same harmonic amplitudes but differing in their initial phase angles, have the same autocorrelation function. Thus, the power spectrum of a periodic function is independent of the phase angles of its harmonics.

2.3.7.2 Determination of Autocorrelation Function

One method of obtaining a correlation analysis by analog means is illustrated in Figure 37. An almost identical arrangement is used for both autocorrelation and crosscorrelation analysis. For autocorrelation, a stored signal is applied to a multiplier both directly and through a variable time delay (τ). For crosscorrelation, two separate signals are applied to the multiplier, one delayed τ seconds with respect to the other. A plot of normalized integrator output versus time delay is the autocorrelation function.

If the signal is available in digital form, a standard digital computer can be programmed to rapidly perform correlation analysis.

2.3.7.3 Crosscorrelation Function

The crosscorelation function for two distinct time varying parameters is defined as:

$$R_{12}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_1(t)\, x_2(t+\tau)\, dt \tag{121}$$

and is again a function of τ. $|R_{12}(\tau)|$ is then a measure of the correlation between $x_1(t)$ and $x_2(t)$ when observed at any two times separated by the time difference τ. Note that $R_{12}(\tau) = R_{21}(-\tau)$ and that $R_{12}(\pm\infty) = 0$. The Fourier transforms relating the crosscorrelation function to the cross-power density spectrum are defined as:

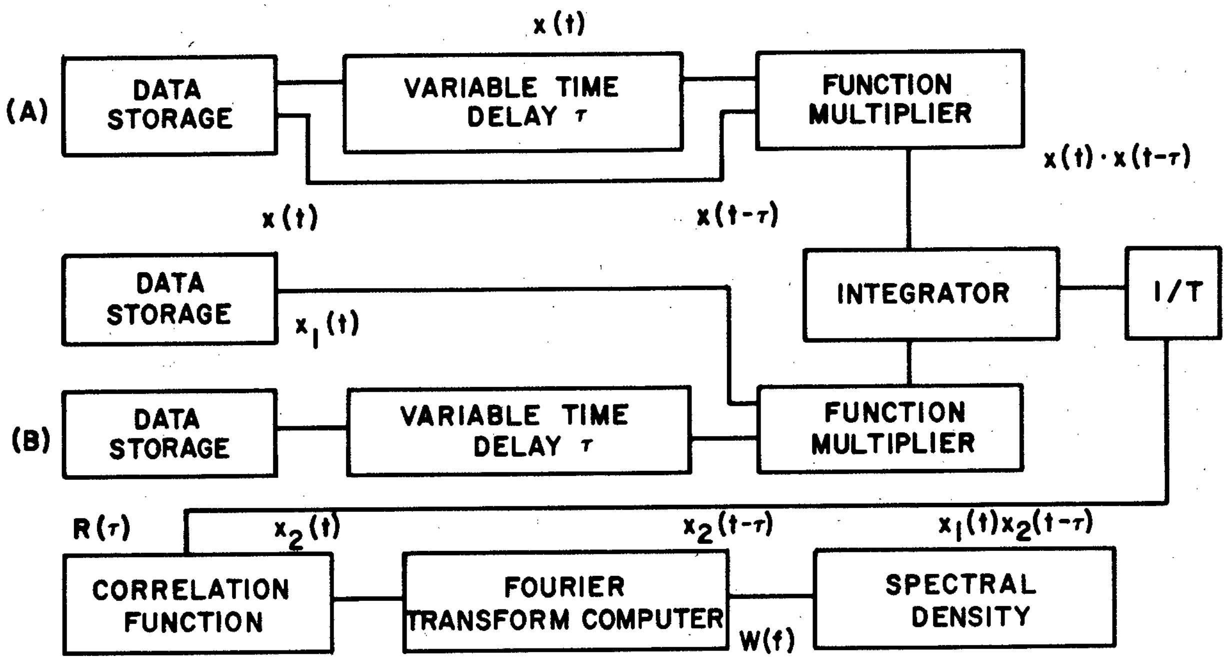


Figure 37. Correlation Analysis.

$$W_{12}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_{12}(\tau)\epsilon^{-j\omega\tau}\, d\tau \qquad (122)$$

and

$$R_{12}(\tau) = \int_{-\infty}^{+\infty} W_{12}(\omega)\epsilon^{j\omega\tau}\, d\omega \qquad (123)$$

where $W_{12}(\omega)$ is the cross-power density spectrum of the random functions $x_1(t)$ and $x_2(t)$.

2.3.7.5 Determination of Crosscorrelation Functions

A method of computing crosscorrelation functions by analog means is shown in B of Figure 37. Digital computation techniques can be used to good advantage in the calculation of crosscorrelation. In either method the signal $x_1(t + \tau)$ used previously for autocorrelation analysis, is replaced by $x_2(t + \tau)$.

3.0 CONCEPTS IN SHOCK DATA ANALYSIS

In connection with environmental testing or structural evaluations, shock may be defined as a pulse, (transient or change of acceleration) which stands out distinctly above the general vibration level and terminates in a time that is short by comparison with the shortest decay time associated with the structure which produced the data.

3.1 Engineering Uses of Shock Measurements

In general, a shock measurement in the form of a time-history of the acceleration or some other parameter is not the most useful type of presentation upon which to base an engineering evaluation. The measured shock often includes the vibrational response of the structural element to which the shock measuring device is attached. This vibration could obscure the determination of the actual shock phenomena whose presentation is desired. Data reduction, in this case, could eliminate many irrelevancies of the measured data. What is most often desired is a presentation from which theoretically predicted values can be compared with the measured results or else a presentation which can be used for the calculation of structural response.

3.1.1 Concepts of Shock Data Reduction

One of two basic methods normally is employed to describe shock. It can be described in terms of the inherent properties of the shock, either in the time domain or the frequency domain, or a description can be made of the shock in terms of its effect on structures when the shock acts as the excitation. The applicable method will be determined by the intended usage of the reduced data. Regardless of the type of reduction employed, a record of the original time-history should be included with the data whenever possible.

3.1.2 Reduction to the Frequency Domain

Any transient function, such as a pulse, can be broken down into a continuous Fourier spectrum. If the transient is recorded on a tape loop and played back continuously, a good approximation of the Fourier spectrum can be obtained with almost any wave form analyzer. The time functions and resulting spectrum functions for a dozen representative transients are given in Figures 38 through 40. Graphical analysis for the determination of Fourier coefficients can often be employed successfully, however, the time involved in graphical determination as well as the availability of short transient automatic recording and analysis devices and digital computers, usually results in a preference for electronic analysis.

The description of a signal by its Fourier spectrum is most often associated with linear system analysis. In the reduction and usage of shock data, if the properties of the structure upon which the shock acts are defined as a function of frequency (the structure is often assumed linear over some

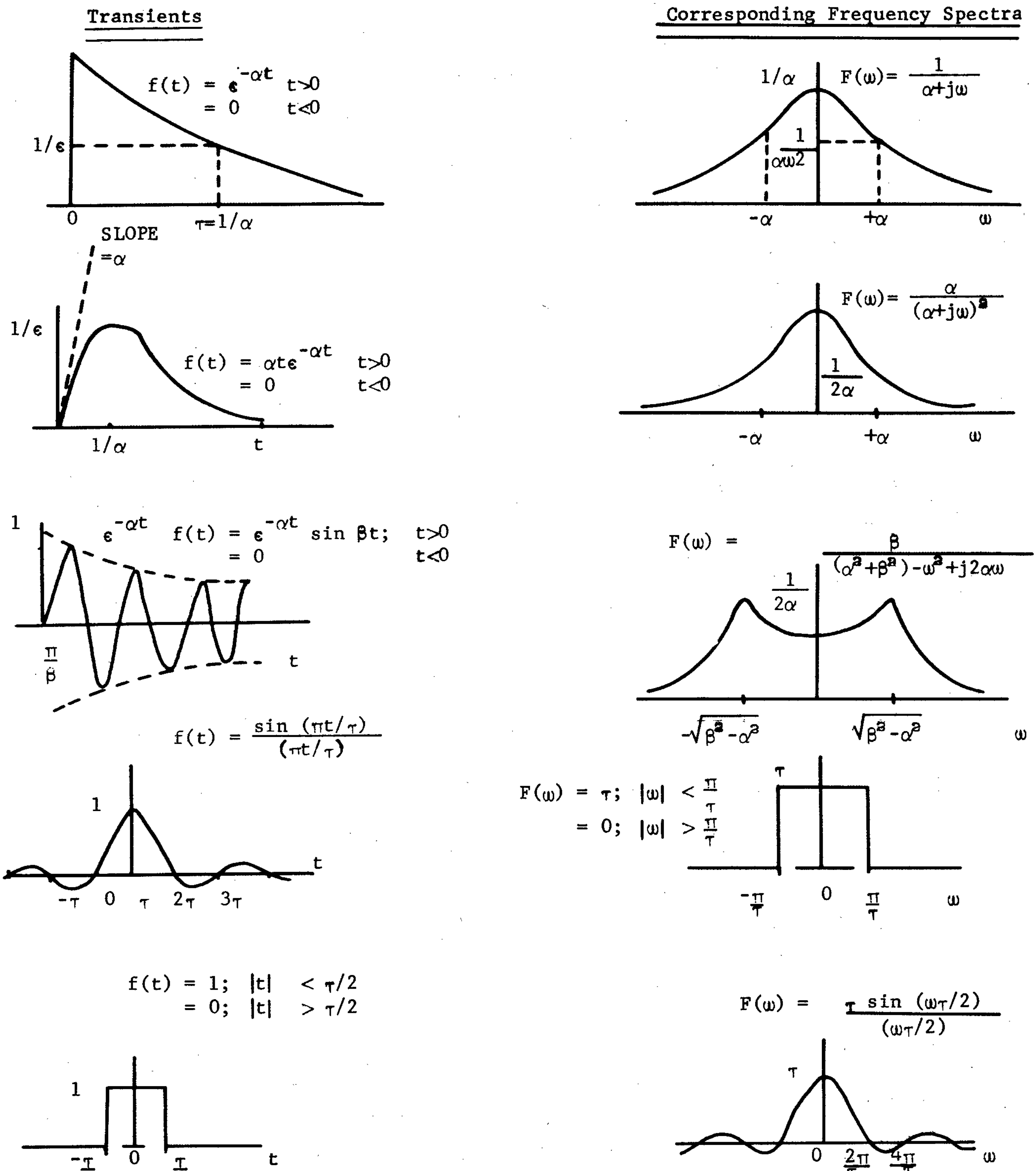


Figure 38. Typical Transient Waveforms and Corresponding Frequency Spectra.

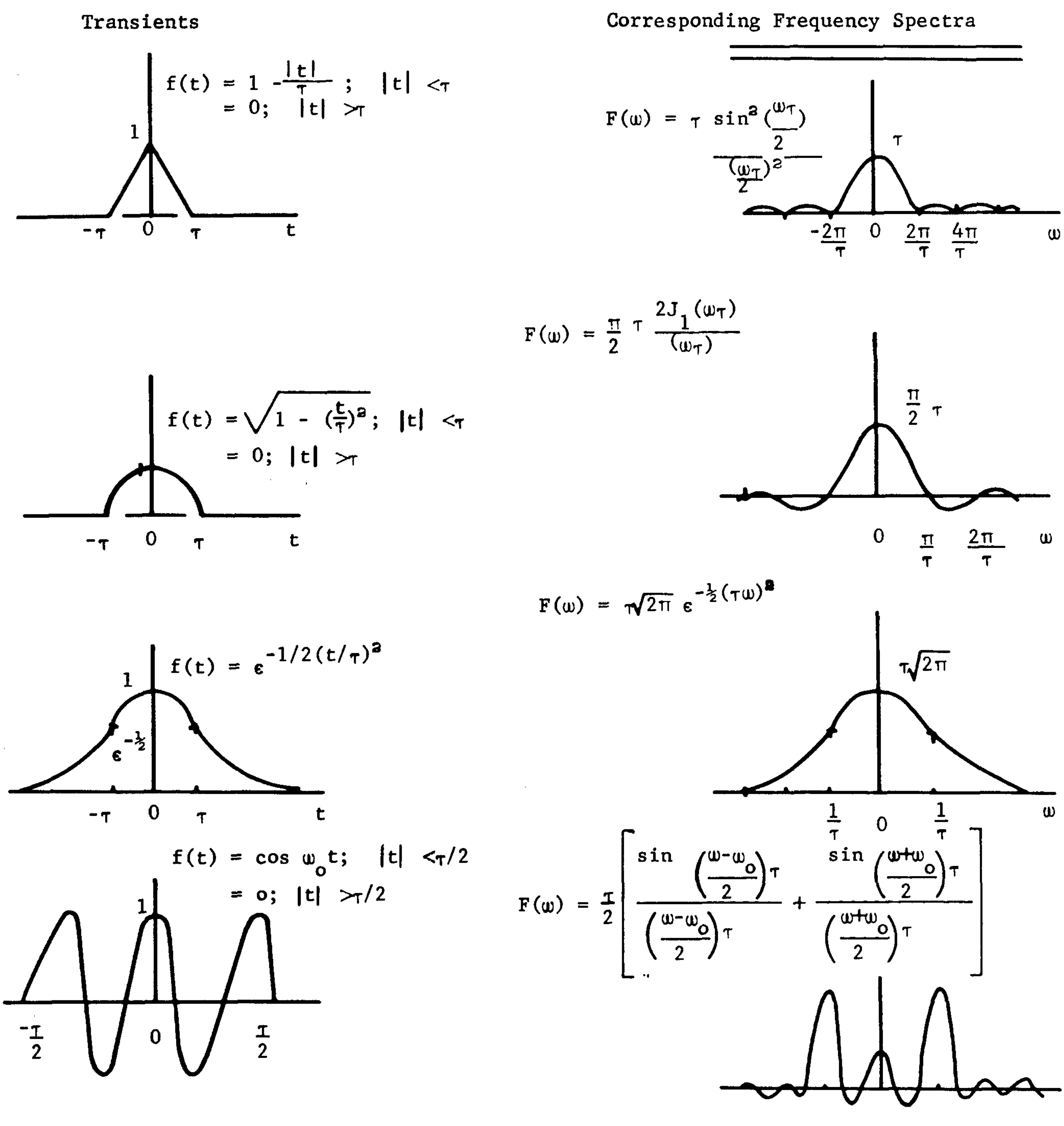


Figure 39. Typical Transient Waveforms and Corresponding Frequency Spectra.

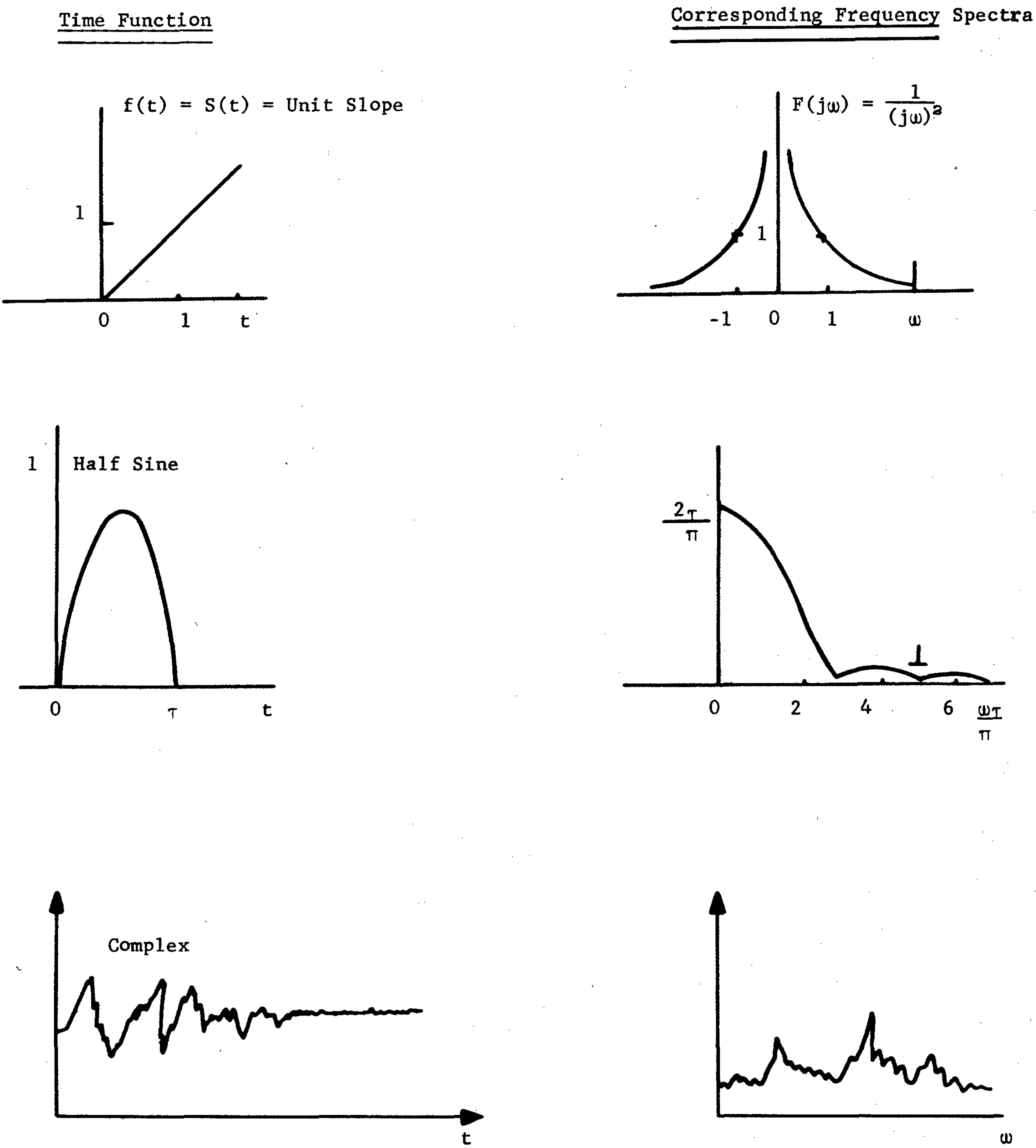


Figure 40. Typical Time Function and Corresponding Frequency Spectra.

range of parameters) they are generally called transfer functions. Commonly used transfer functions in shock and vibration work include mechanical impedance, mobility, and transmissibility.

The transfer function of a structure can sometimes be determined by applying a pulse of force to the structure and observing its response. If such a transfer function is determined to be $T(\omega)$ then the Fourier spectrum of the response, $R(\omega)$, is the product of the Fourier spectrum of the shock $F(\omega)$ and the transfer function $T(\omega)$.

3.1.3 Reduction to the Response Domain

When a shock is applied to a structure, a response characteristic of the structure ordinarily is produced. If certain assumptions concerning the properties of the structure are made, the response can be assumed to depend only on the damping of the structure. The magnitudes of the peaks of the response can then be systematically related to the properties of the system. The description of a shock in terms of its Fourier spectrum differs from this in two ways:

a. If a knowledge of the peak responses of a structure to a shock excitation exists, the time-history of the shock still cannot be determined on this basis alone; i.e., the calculation of the peak responses is an irreversible process.

b. The response spectrum describes the effect of a shock upon a structure in terms of its peak responses where, on the other hand, the Fourier spectrum defines a shock in terms of the amplitudes and phases of its frequency components.

If a representation of the peak responses of many simple structures is obtained from the shock measurement, the process is called reduction to the response domain. For a system that responds in a single degree of freedom, this type of reduction applies directly. However, for a system that responds in more than one degree of freedom, this method of reduction is applicable only if the principle of superposition can be used with validity.

3.1.4 The Shock Spectrum

The shock spectrum for a given shock is defined as the peak acceleration response of a simple undamped resonant member to this shock, plotted as a function of the frequency to which the member might be tuned. It thus serves as an indication of possible stress within the component, whereas the peak acceleration of the original complex serves merely as an indication of stress at the mounting points of the component.

In common terminology, the term shock spectrum is used synonymously with two-dimensional shock spectrum and represents the maximum value of the response found in a single time-history. The three-dimensional shock spectrum is also encountered and takes the form of a surface which shows the distribution of response peaks throughout the time-history.

3.1.4.1 Relation Between Shock Spectrum and Fourier Spectrum

If $F(\omega)$ is the Fourier spectrum of a pulse where:

$$F(\omega) = \int_{-\infty}^{+\infty} F(t)\epsilon^{-j\omega t}\,dt \tag{124}$$

then for a pulse starting at t equals 0, it is:

$$F(\omega) = \int_{0}^{\infty} f(t)\epsilon^{-j\omega t}\,dt \tag{125}$$

It can be shown that a quantitative relationship exists between the Fourier spectrum of absolute values and the peak residual response of an undamped simple structure.

If $F(\omega_n)$ is the Fourier amplitude spectrum or the absolute value of $F(\omega)$ at the frequency ω_n, then an equivalent static acceleration (A eq.) can be assigned to the maximum residual displacement response. If the fraction of critical damping of the simple structure used for response measurements is zero then:

$$A\ eq. = \frac{\omega_n}{g} F(\omega_n) \tag{126}$$

where g is the acceleration due to gravity.

3.1.5 Methods of Data Reduction

Generally, in shock data reduction, the measured shock is not definable by an analytic function but is available in graphic or electrical form. Several reduction methods have been described in preceding paragraphs. However, a general summation of available methods of shock data reduction would have to include reduction to the Fourier spectrum (by wave analysis, digital computation, use of trigonometric forms, and tables) and reduction to the shock spectrum (by graphic methods, numerical methods, mechanical, electrical analog methods, and use of the reed gauge).

4.0 CONCEPTS OF ACOUSTICAL DATA ANALYSIS

4.1 General

A qualitative measure of the relative "loudness" of noise, or some other sound, is often made in connection with the testing and evaluation of jet aircraft and missile systems. This measurement can then be used for the evaluation of personnel safety, system performance, or specification design. With the recent introduction of large, reverberant test chambers and sophisticated equalization equipment, acoustical testing can conceivably be performed directly from the raw data, much in the same way that inflight vibration data can conceivably be faithfully reproduced on a shaker table. Thus, in vibration and acoustics, a measure of the amplitude of the environment as well as its frequency spectrum will serve many useful purposes.

4.1.1 Units of Measure

In the description of the loudness of sound many, sometimes confusing, terms are used and quite often coincidentally. These terms include sones, phons, decibels, spectrum level, and the relative term - loudness.

So that a scale of loudness as a function of the sound pressure level in db can be obtained, subjective measurements are made with a pure tone. The frequency of this tone is normally 1000 cps and a reference level is chosen at 40 db referenced to (re) 2×10^{4} μ bar and is equal to one sone or 1000 millisones. The intensity of sound normally is expressed in terms of sound pressure in μ bars and is equal to one dyne per centimeter squared (dyne/cm^2). A loudness of 10 sones is then ten times as loud as 1 sone. The relationship between loudness in millisones and loudness level in phons has been determined empirically and produces a curve which shows that at 1000 cps, the loudness level in phone corresponds to the sound pressure level in db re 2×10^{4} μ bar. The dependence of loudness level (phons) on frequency has been established. Thus, loudness in sones can be applied to sound of any frequency or combination of frequencies. As an example of the nonlinearities involved, a reduction in loudness level from 70 phons (10,000 millisones) to 40 phons (1000 millisones) corresponds to a tenfold change in loudness. But a change from 40 phons to 20 phons is again a tenfold change in loudness to 100 millisones.

4.1.2 Addition of Loudness Levels

Because of phasing, masking, and other considerations, loudness in general is not directly additive. Since, however, the frequency components of a given noise are often determined by filtering the overall noise, it has been found convenient to express total loudness of data passed through nth octave bandpass filters. This results in the equation:

$$S_t = S_m + F(\Sigma s - S_m) \tag{127}$$

Where S_t is the total loudness in sones, S_m is the loudness in sones of the loudest nth octave band, and Σs is the sum of the loudness in sones of all the nth octave bands. By taking the bandwidth and masking effect into account, the following F factors were obtained:

a. 1/3 octave bandwidth where F = 0.15
b. 1/2 octave bandwidth where F = 0.2
c. 1 octave bandwidth where F = 0.3

This is also shown graphically in Figure 41.

In addition to equal loudness contours, the concept of equal noisiness contours has also received much attention and gives a comparable answer in the calculation of loudness level of a specific sound.

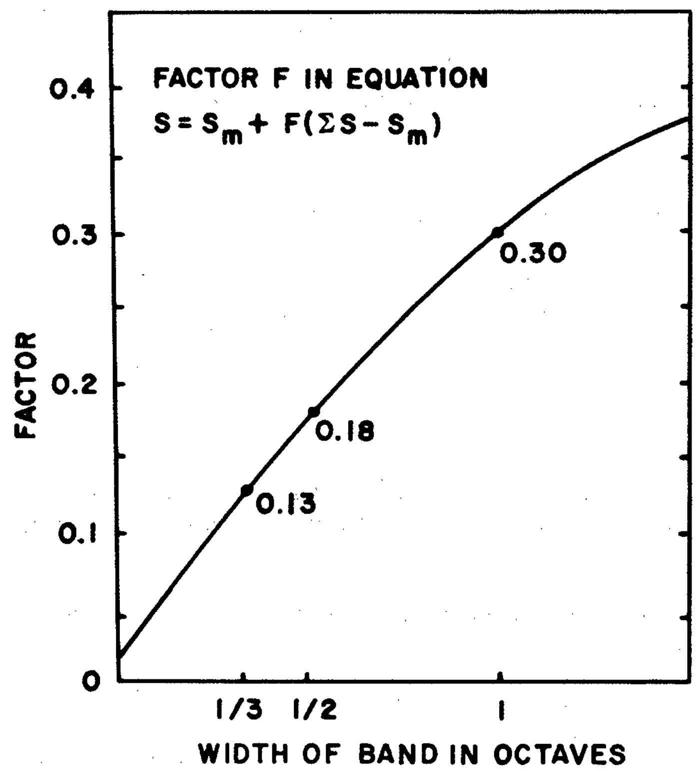


Figure 41. Value of F for Bands of Various Widths.

4.1.3 Pure Tones - Discrete Frequencies

A pure tone in acoustical data analysis is analogous to a vibrational acceleration at a discrete frequency and both result in a line spectrum. Sounds with line spectra can be analyzed with any type of analyzer which has sufficient resolution. The presentation of the data may be either in tabular form (spectral linesversus frequency) or a plot in the form of a bar diagram with either a linear or logarithmic base. A logarithmic scale is preferable if the sound under analysis contains widely spaced components. If it contains a number of components within a relatively selective (narrow) frequency range, a linear scale is preferable.

A sound whose analysis produces many discrete spectral lines or bars is often called a "noise with a line spectrum" as opposed to a pure tone of single frequency and one spectral line.

4.1.4 Complex Noise

Sounds with continuous spectra (that is, noises which display no prominent components at discrete frequencies) are generally analyzed with either constant bandwidth or constant percentage bandwidth (constant Q) filter analyzers. To be able to compare the results as obtained by different analyzers, the observed data should be reduced to spectrum levels. The spectrum level is defined as the level in db of the effective sound pressure in bands one cycle wide. This is entirely analogous to the concept g^2/cps where the mean squared acceleration was divided by the filter bandwidth, to reduce instrumentation effects.

To reduce a measured level L db taken by a filter band with a width of Δf cps to a spectrum level of S db, the following operation is performed.

$$S = L - 10 \log_{10} \Delta f \text{ db} \qquad (128)$$

This spectrum level S is then plotted at the geometric mean frequency of f_m equal to $\sqrt{f_1 \ f_2}$ of the band where f_1 and f_2 are the nominal cutoff frequencies.

The function $10 \log_{10} \Delta f$ will be a constant for a heterodyne type (fixed bandwidth filter) analyzer and for a constant percentage bandwidth type, it will increase in magnitude by three db for each doubling of frequency.

Generally, all measurements of continuous spectra noise should be reduced to spectrum level.

4.1.5 Instruments for Analyzing Noise

In previous sections, various references have been made to filter sets, analyzers, integrators and plots of analyzed data. Several types of waveform analyzers and their important characteristics will not be enumerated; some normally-used readout mechanisms will also be included. These instruments are as follows:

a. Octave Band Analyzer In an octave band analyzer, both the relationship between adjacent filter center frequencies and between the upper and lower 3-db points of one particular filter is 2 to 1. (Two frequencies are an nth octave apart if $\frac{f_2}{f_1}$ equals 2^n.) A standardized set of octave

passbands, commonly used in the United States has the following passbands: 75 to 150, 150 to 300, 300 to 600, 600 to 1200, 1200 to 2400, and 2400 to 4800 cps. Another commonly used series of octave bands is 50 to 100, 100 to 200, ... 3200 to 6400 cps. Besides the 2 to 1 relationship between center frequencies and cutoff frequencies, the other useful relationship in connection with octave band analysis is that the 3-db passband of any arbitrary, constant percentage, octave filter is 0.67 times the filter center frequency.

b. Half-Octave and Third-Octave Analyzers If a more detailed analysis is required, a constant percentage bandwidth analyzer with narrower fixed filters can be used. These filter bands can be a half or a third of an octave in width or even narrower. The one-third octave analyzers in use are normally designed on the basis of the "preferred numbers". A design on the preferred number basis selects center frequencies whose common logarithm has a mantissa which is an even tenth. For example, .1000, .2000, .3000 are the mantissas respectively for 1.26, 1.58 and 1.99. These antilogs then would be rounded off to 1.25, 1.6 and 2.0 respectively, and they, as well as the series formed by their multiplication with integral powers of ten, would form the basis for filter center frequencies. Similarly, those numbers whose common log mantissa is .1500, .2500, .3500, etc., form the basis for 3-db crossover points between filters. The third-octave relationships, namely a factor of 1.25 between adjacent center frequencies, and a 3-db bandwidth-to-center-frequency ratio of 0.231 then follow from the fact that one-third octave is equal to one-tenth decade.

Half-octave relationships reduce to 1.414 for the adjacent center frequency ratio and 0.34 for the bandwidth-to-center-frequency ratio. (See Figure 42.)

c. Narrow Band Proportional Bandwidth Analyzers Analyzers which have passbands even narrower than one-third octave are also used, to determine more exactly the presence of discrete components in a signal. This type of analyzer normally is continuously tunable.

d. Constant Bandwidth Analyzers Many different models of constant bandwidth analyzers are now available. Generally, all the models are similar in one respect, i.e., they first combine the input signal with a tunable local oscillator in a modulator circuit, and then filter the result using a constant bandwidth filter. The filtering can take place at a fixed IF (for example, at 100 kc) or at d-c (zero IF). If a fixed IF is employed, crystal filtering is often used, whereas in the zero IF schemes, Chebychev or simple resistance-capacitance (RC) filtering normally is encountered. Constant bandwidths as narrow as 1/4 cps operating to 30 kc and higher have been successfully used in this type analyzer. Other features often offered in this regard include: various filter bandwidths, selectable by the operator; optional detection of the filtered signal (linear, peak, or square law); automatic normalization (division by filter bandwidths); linear or logarithmic sweep of frequency versus time; mechanical or all electronic sweep drives; optional smoothing of output data; digital output of analyzed data and other optional display features.

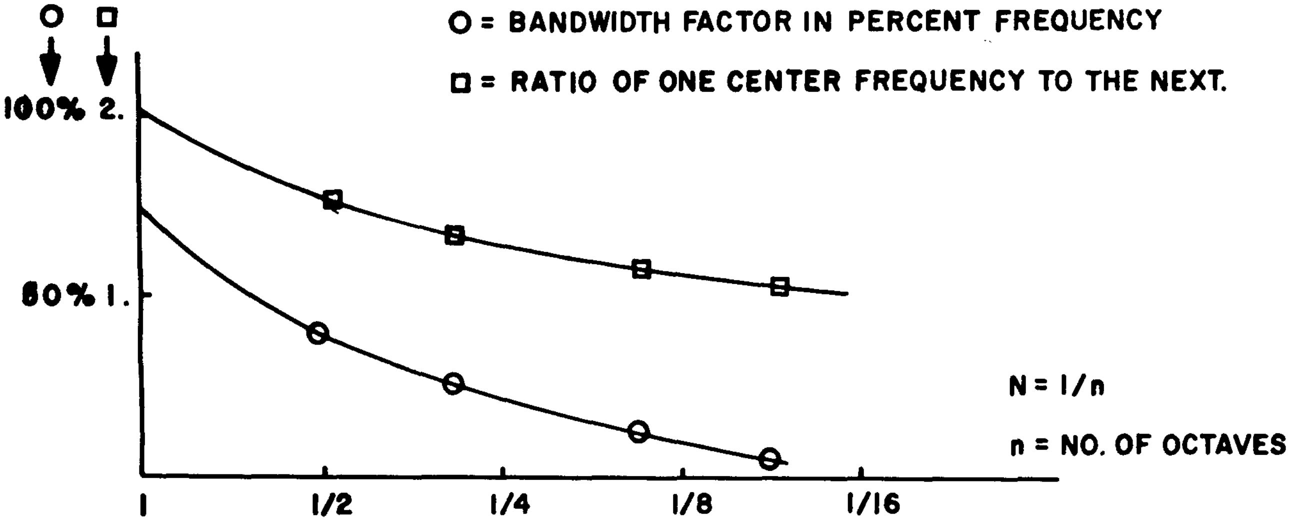


Figure 42. Filter Relationships (nth Octave).

e. Parallel Filter Analyzer - Parallel Readout The constant percentage bandwidth analyzers discussed so far, are examples of parallel filter analyzers which operate over only one selectable passband at a time. Other parallel filter analyzers which provide "simultaneous" readout from many filters are also available. One such device uses 480 parallel magnetostrictive filters of constant bandwidth and a rapidly commutated output which can be observed on an oscilloscope or recorded on a strip chart.

Another method uses parallel constant Q filters, variable detection digital averaging, and a commutated output. These analyzers have the advantage of presenting an almost continuous output over the entire spectrum of interest and so give an indication of "real time" variation of the input data spectrum.

f. Circulating Memory Analyzers A significant contribution to the development of spectrum analysis has been achieved recently by the development of the circulating delay line analyzer. At least two different analyzers of this general type are presently available. One of these analyzers, heterodynes the incoming signal to a suitable frequency range where it is circulated up to 360 times a second in a crystal delay line. By a unique phase-advance arrangement, a highly selective analysis is performed with near optimum signal-to-noise enhancement and a visual display of spectrum coefficients is presented every second. In this way spectral components over a fixed range of frequencies are displayed with an accuracy of nearly one cps in essentially real time.

Another analyzer of this type samples and stores the incoming data in several parallel delay lines. By using delay line, time-compression techniques, a speed of up to 50,000: 1 is achieved and a frequency resolution of 0.01 cps, or more, is available. A digital output as well as standard analog displays is readily available from this analyzer.

Besides having the advantages of resolution and speed, these analyzers do not present the filter storage or ringing problems normally encountered. Unless rapid observation of large amounts of data is necessary, the relative costs of this type analyzer might make them impractical.

4.1.6 Commonly Used Readout Devices

The most commonly used readout devices in data analyzers are as follows:

a. Level Recorders High Speed level recorders are often used to display filter outputs. Linear or logarithmic ordinate scales are normally available. With this type of presentation, the time dependence of data through a fixed filter bandwidth can be observed.

b. Oscillograph Recorders An oscillograph record is the most straightforward means to display the total unfiltered signal. Frequently, on one record, are displayed not only the total unfiltered signals, but the outputs of several filters simultaneously or the differentiated or integrated data as well. Averaging or smoothing or even rectification of the filter output can be done before recording.

c. Oscilloscope Oscilloscope displays have been used extensively with "real time" analysis systems. This has the advantage of preserving, (through the use of photographic equipment) both the frequency and the time dependence of the data, since a display of amplitude versus frequency is given every display period. (Display periods presently range from approximately every 1/2 second to every two seconds. In some cases, continuous display of frequency-amplitude information is given.) The rapid sampling and buffer storage necessary to convert these visual displays to a permanent record (other than a photograph) makes the conversion involved and sometimes expensive.

d. X-Y Recorders In most cases where a time average presentation of frequency-amplitude information is required, an X-Y plot is suitable. In probability density analysis, this method has been used almost exclusively. It can be extended for use with an oscilloscope presentation if the process of sampling, conversion, and tabulation is previously undertaken.

5.0 ANALYSIS OF STRAIN DATA

5.1 STRAIN IN ONE DIRECTION

It recently has been found that the analysis of stress and strain applied to structurally loaded systems, can often detect and sometimes lead directly to solution of design or application problems relating to structural failure. If a simple structure is involved and the direction in which a strain measurement is required is known, then one gauge will give a close indication of true strain, "at a point". (The manner in which strain is measured with a gauge gives the average over a small distance; this normally presents no problem unless the strain gradient is extremely large. If, then,

a discrete value can be assigned to the material under study as its modulus of elasticity, stress can be calculated. Quite often however, because of the complexity of the structure to be analyzed, little or no prior knowledge of directions of principle stresses is available. In such cases, a strain gauge rosette is the best approach.

5.2 STRAIN GAUGE ROSETTES

To completely specify the state of stress at any point on a free surface, three independent quantities must be known. These quantities are the magnitude of the two principle stresses, σ_1 and σ_2, and their directions with respect to some reference. Generally, it will be necessary to make three independent measurements of strain to determine these three quantities.

The three most commonly encountered strain rosette patterns are shown in Figure 43. By means of the three or four measurements indicated, the strain in any arbitrary direction, with respect to a fixed reference, may be obtained from application of Mohr's strain circle or Mohr's circle of stress.

The equations which describe the principle stresses, the maximum shearing stress, and the direction of principle stresses in terms of measurement strains are as follows:

a. For the rectangular strain rosette:

$$\sigma_1 = \frac{E}{2}\left[\frac{e_a + e_c}{1 - \mu} + \frac{1}{1 + \mu}\sqrt{2(e_a - e_b)^2 + 2(e_b - e_c)^2}\right] \tag{129}$$

$$\sigma_2 = \frac{E}{2}\left[\frac{e_a + e_c}{1 - \mu} - \frac{1}{1 + \mu}\sqrt{2(e_a - e_b)^2 + 2(e_b - e_c)^2}\right] \tag{130}$$

$$\tau_{max} = G\sqrt{2(e_a - e_b)^2 + 2(e_b - e_c)^2} \tag{131}$$

$$\phi = \tfrac{1}{2}\tan^{-1} \pm \left[\frac{(2e_b - e_a - e_c)}{(e_a - e_c)}\right] \tag{132}$$

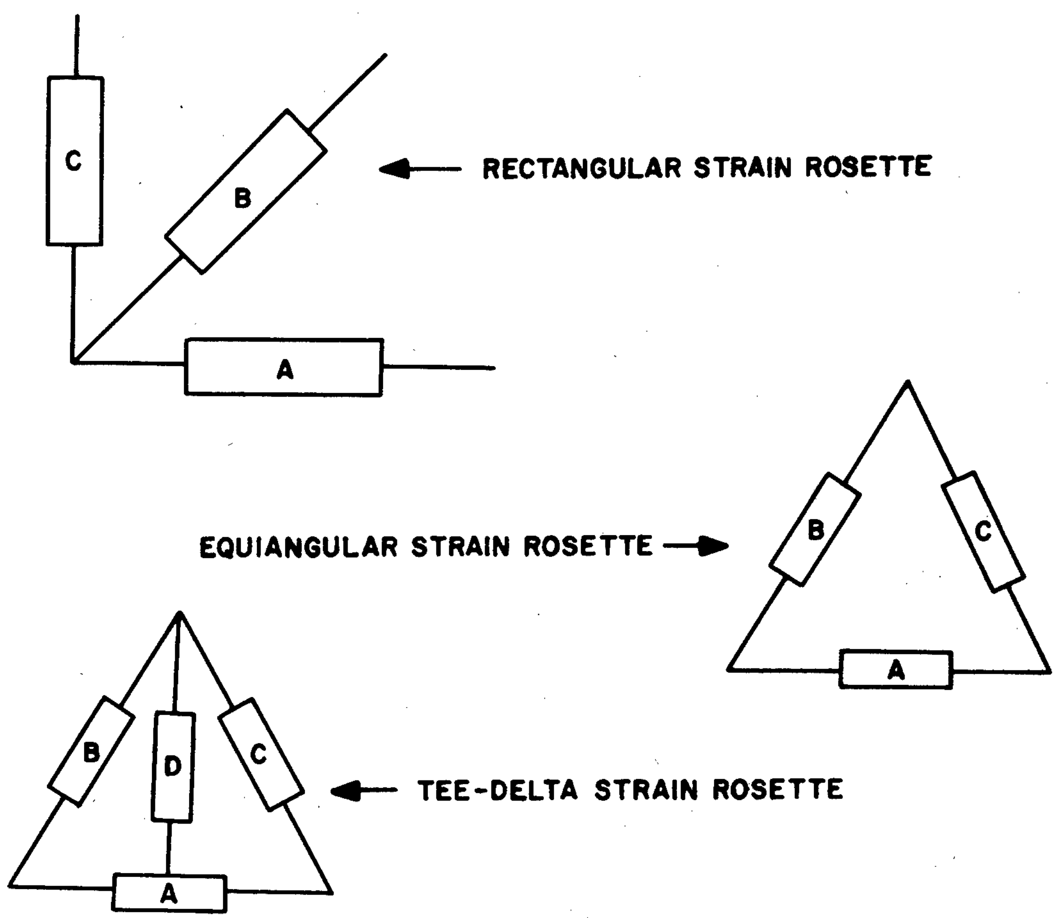


Figure 43. Strain Rosette Patterns.

b. For the equiangular strain rosette:

$$\sigma_1 = E\left[\frac{e_a+e_b+e_c}{3(1-\mu)} + \frac{1}{1+\mu}\sqrt{\left(e_a - \frac{e_a+e_b+e_c}{3}\right)^2 + \frac{(e_c-e_b)^2}{\sqrt{3}}}\right] \quad (133)$$

$$\sigma_2 = E\left[\frac{e_a+e_b+e_c}{3(1-\mu)} - \frac{1}{1+\mu}\sqrt{\left(e_a - \frac{e_a+e_b+e_c}{3}\right)^2 + \frac{(e_c-e_b)^2}{\sqrt{3}}}\right] \quad (134)$$

$$\tau_{max} = 2G\sqrt{\left(e_a - \frac{e_a+e_b+e_c}{3}\right)^2 + \left(\frac{e_c-e_b}{\sqrt{3}}\right)^2} \quad (135)$$

$$\phi = \frac{1}{2}\tan^{-1}\left[\frac{\frac{1}{\sqrt{3}}(e_c-e_b)}{e_a - \frac{e_a+e_b+e_c}{3}}\right] \quad (136)$$

c. For the tee-delta strain rosette:

$$\sigma_1 = \frac{E}{2}\left[\frac{e_a+e_d}{1-\mu} + \frac{1}{1+\mu}\sqrt{(e_a-e_d)^2 + \frac{4}{3}(e_b-e_c)^2}\right] \quad (137)$$

$$\sigma_2 = \frac{E}{2}\left[\frac{e_a+e_d}{1-\mu} - \frac{1}{1+\mu}\sqrt{(e_a-e_d)^2 + \frac{4}{3}(e_b-e_c)^2}\right] \quad (138)$$

$$\tau_{max} = G\sqrt{(e_a - e_d)^2 + \frac{4}{3}(e_b - e_c)^2} \tag{139}$$

$$\phi = \frac{1}{2}\tan^{-1}\frac{2}{\sqrt{3}}\left[\frac{e_b - e_c}{e_a - e_d}\right] \tag{140}$$

where σ_1 and σ_2 are the principle stress magnitudes, τ_{max} is the maximum shearing stress, ϕ is the direction of the principle stresses, μ is Poisson's ratio, E is the modulus of elasticity in shear and e_a, e_b, e_c, and e_d are the measured strains.

Multiple gauge connections can be used to measure such parameters as thrust on a circular shift, transverse shear on a member in bending, torque on a circular shaft, and bending moment or torque at inconvenient or inaccessible locations.

5.3 PRESENTATION OF STRAIN DATA

Dynamic strain data, associated with the structural evaluation of missile systems, can be presented in various ways. For example, if a launching rail were instrumented with several strain gauges, the stress distribution versus time, along the rail, could be presented in tabular or pictorial form. Or, if an oscillatory motion is encountered, as is often the case, the number of cycles during which a preselected strain level is encountered could be presented. Such information would be useful in fatigue analysis.

Since a spectrum analysis of strain data normally is not required, oscillograph recordings of strain information in most cases are sufficient. From these recordings, such values as maximum strain and short term average strain can be determined as well as correlation with other significant events. The recorded data can be sampled and converted into discrete numbers. In this manner, general purpose digital computers have been successfully programmed to evaluate such parameters as those in equations (139) to (140).

GLOSSARY

1. Acceleration of Gravity (g): The acceleration of gravity is taken as $32.2\ ft/sec^2$, $386\ in./sec^2$, or $980.7\ cm/sec^2$. The nondimensional unit of acceleration, G, is defined as a/g, which equals the number of gravity units. The acceleration is a.

2. Amplitude: In vibration terminology, the word amplitude is usually reserved to represent the maximum value of a sinusoidal vibration. If used in other senses, its meaning should be carefully indicated.

3. Bandwidth, or Passband Width: Electrical filters can be constructed to pass signals between two frequency limits and to reject signals outside these limits. The difference between these frequencies is the bandwidth, the average of these two frequencies is the center frequency. Octave bands, half-octave bands, etc., are commonly used. Smaller bandwidths are required if good spectral resolution is wanted.

4. Conversion from One Band Pressure Level to Another: If the sound pressure level is determined as p_1 for a bandwidth Δf_1, then the band pressure level for a bandwidth Δf_2 can be calculated as:

$$L_{p\Delta f_2} = 10 \log \left[\frac{p_1^2}{p_o^2} \cdot \frac{\Delta f_2}{\Delta f_1}\right]$$

$$= 20 \log \frac{p_1}{p_o} - 10 \log \frac{\Delta f_1}{\Delta f_2}$$

$$= Lp_1 - 10 \log \frac{\Delta f_1}{\Delta f_2}$$

It has been assumed for the above that the spectral level is constant throughout the frequency band.

5. Decibel (db): The decibel is a unit of level which determines the ratio of two quantities that are proportional to power; the number of decibels is equal to 10 times the logarithm to the base 10 of this ratio.

6. Discrete Frequency: A vibration of discrete frequency is a sinusoid.

7. Distribution Curve: The ordinate of the distribution is called probability density and the abscissa is the deviation of the magnitude of the function from its mean value. If the function represents a vibration, and if the mean value of the vibration is zero, the deviation is equal to the instantaneous amplitude of the vibration. The area under the distribution

curve, and bounded by two abscissa values, is equal to the probability that the instantaneous amplitude of the vibration will be contained between the two abscissa values. This area can also be considered as the fractional part of the time that the instantaneous amplitudes of a vibration are within this range.

8. Double Amplitude: Double amplitude may be used for sinusoidal vibrations as equal to two times the amplitude. This term has come into use because it is the excursion visually observed in low-frequency vibrations.

9. Gaussian (or Normal) Distribution: The probability density for the instantaneous amplitudes for the usual type of random vibrations is approximately given by the Gaussian (or normal) distribution curve.

10. Half-Sine Shock Pulse: A shock excitation for which the acceleration time curve has the shape of the positive (or negative) phase of one cycle of a sine wave.

11. Intensity Level: The intensity level of a sound, in decibels, is 10 times the logarithm to the base 10 of the ratio of the intensity I of this sound to the reference intensity I_o. The reference intensity I_o shall be stated. A generally used value, especially for air acoustics is 10^{16} watts/cm^2.

Note

> Conventional sound pressure meters and sound level meters do not measure intensity. Hence, the words "intensity level" should not be applied to data taken with them. Instead, the two expressions "sound level" and "sound pressure level" are used in this MTP as follows: (a) Sound level is applied to data taken on instruments which meet the specifications for sound level meters drawn up by the American Standards Association. (b) Sound pressure level is applied to data taken by a sound pressure meter with a flat response. In both cases the reference pressure is 0.0002 dyne/cm^2.

12. Instantaneous Amplitude: The magnitude of a vibrational motion at any given instant of time.

13. Level: In communication and acoustics, the level of a quantity is the logarithm of the ratio of that quantity to a reference quantity of the same kind. The base of the logarithm (usually 10), the reference quantity, and the kind of level must be specified.

14. Line Spectra (Spectrum): A line spectrum is an assembly of discrete frequencies plotted amplitudes.

15. Loudness Level: The loudness level of a sound is numerically equal to the sound pressure level in decibels of the 1000-cycle pure tone which is judged by the listeners to be equivalent in loudness. The 1000-cycle comparison tone shall be considered as a plane sinusoidal sound wave coming from a position directly in front of the observer. The listening is to be done with both ears, and the intensity level of the 1000-cycle comparison tone is to be measured in the free progressive wave. The reference sound pressure is 0.000200 dyne/cm^2. The unit is the phon.

16. Masking: The number of decibels by which a listener's threshold of audibility for a given tone is raised by the presence of another sound.

17. Mean Square Acceleration Spectral Density: Also known as power density, acceleration density, acceleration spectral density or spectral density. The term "acceleration density" is most frequently used. (Velocity displacement, or other appropriate terms may be substituted for acceleration). As defined for random vibration, the mean square acceleration spectral density, or the acceleration density, is the mean of the square of the acceleration amplitude per unit bandpass width; i.e., it is the limiting value of the mean square acceleration passed by a rectangular bandpass filter, divided by the bandwidth, as the bandwidth approaches zero.

18. Mean Square Acceleration Density Spectrum: Spectral density expressed, or plotted, as a function of frequency constitutes a density spectrum. The mean square acceleration density spectrum is abbreviated to power spectrum, density spectrum, or acceleration density spectrum. The last term appears to have most general acceptance.

19. Microbar (dyne per square centimeter): The unit of sound pressure; one microbar equals 1 dyne/cm^2.

20. Noise: In acoustics, noise is defined as unwanted sound. In vibration terminology, noise refers to vibrations which are nonperiodic and of a generally random nature. This latter concept also applies to background electrical disturbances that are characterized as noise.

21. Octave: The interval between any two tones whose frequency ratio is 2:1.

22. Peak-to-Peak Amplitude: While the term usually is synonymous with double amplitude, it may represent a different value for periodic complex waves where it represents the sum of the absolute values of the maximum positive and negative magnitudes.

23. Phon: The unit for measuring the loudness level of a tone. The number of phons is equal to the number of decibels a 1000-cycle tone is above the reference sound pressure when judged equal in loudness to the tone in question.

24. Power Spectrum Level: The power spectrum level is equivalent to the spectral density described for mechanical vibrations. It is the acoustic power per unit bandwidth. This is proportional to the pressure-squared

per cps. It is given as:

$$L_{\omega s} = L_{\omega} - 10 \log \Delta f_1$$

25. Pressure Spectrum Level: The effective sound pressure for the sound power contained in a unit bandwidth is the pressure spectrum level. If Δf_2 equals unity, (see equation in definition for "Conversion from One Band Pressure Level to Another), the value of this equation satisfies this definition and pressure spectrum level is seen to be:

$$Lp_s = Lp_1 - 10 \log \Delta f_1$$

26. Probability: Probability is the likelihood of occurrence of a particular event. It is generally estimated as the ratio of the number of occurrences of the particular event to the total number of occurrences of all types of events considered. A certain event has probability one and an impossible event has zero probability. The reverse of these statements is not necessarily true, i.e., an event probability one need not necessarily happen. (See Reference 34.)

27. Probability Density: Probability density is the ratio of the change of probability to the change in number of particular events considered. In random vibration, this is the ratio of the probability that the (instantaneous) amplitude will be within a given incremental range to the size of the increment as the increment approaches zero at a specified amplitude position.

28. Random Vibration: A random vibration is an oscillation whose instantaneous amplitude cannot be specified for anyone given instant of time.

29. Root Mean Square (rms): The rms value of a set of numbers is equal to the square root of the average of the sum of the squares of these numbers. The rms value of a function f(t) between t_1 and t_2 is:

$$\text{rms} = \left[\int_{t_1}^{t_2} \frac{[f(t)]^2}{t_2 - t_1} dt \right]^{\frac{1}{2}}$$

30. Shock Spectrum: A displacement shock spectrum is the distribution of the maximum relative displacement responses of a series of single degree of freedom systems to a shock excitation as a function of the frequencies of the systems. Shock spectra can be expressed in terms of acceleration or velocity by multiplying the displacement spectra by $(2\pi f)^2$ or $2\pi f$, respectively. Spectra can be obtained for either damped or undamped systems. An undamped system is assumed unless otherwise specified.

31. Sone: A unit of loudness. It is defined as the loudness of a 1000-cycle tone 40 db above threshold. A millisone is one-thousandth of a sone and is often called the loudness unit.

32. Sound Intensity: The flow of a sound wave is a passage of sound energy. The power transmitted per unit of area in the direction of travel is defined as the sound intensity and is equal to:

$$I = \frac{p^2}{C}$$

in a free homogenous medium where I is the density, C is the velocity of sound, and p is the pressure.

33. Sound Power: If a sound source emits an amount of acoustic power W, then for a free field the sound intensity at any point a distance r from the source is:

$$I = \frac{W}{4\pi r^2}$$

34. Sound Power Level: The sound power level of a sound source, in db, is 10 times the logarithm to the base 10 of the ratio for the sound power radiated by the source to a reference power. The reference power is usually taken as one micromicrowatt, or 10^{12} watt.

35. Sound Pressure Level: The sound pressure level in db, of a sound is 20 times the logarithm to the base 10 of the ratio of the pressure p of this sound to the reference pressure p_{ref}. The value of p_{ref} should always be stated. A common reference pressure used in connection with hearing and the specification of noise is 0.000200 dyne/cm^2 (or 0.0002 microbar). Another commonly used reference pressure is 1 dyne/cm^2. The two differ exactly 74 db. It is to be noted that in many sound fields the sound pressure ratios are not proportional to the square root of corresponding power (intensity) ratios and, hence, cannot be expressed in db in the strict sense; however, it is common practice to extend the use of the db to these cases.

36. Sound Spectrum Level: The spectrum level is a plot, as a function of frequency, of 20 times the logarithm to the base 10 of the ratio of the effective pressure. The unit is the db.

37. Spectrum: A spectrum of a function of time is a resolution of its components into sinusoidal functions and is expressed as a function of frequency. The spectrum is also used to signify a continuous frequency range of sinusoidal components.

38. Standard Deviation: The rms value of the deviations of a function, or of a set of numbers, from a mean value is the standard deviation. It is

identified by the Greek letter σ.

39. Sweep Rate, Uniform Sweep Rate, Logarithmic Sweep Rate, Constant Octave Sweep Rate: A uniform sweep rate exists if the rate of change of frequency or df/dt is constant. A logarithmic sweep rate exists if the change of frequency per unit of frequency or df/fdt is constant. In this latter case, the time of sweep between any two frequencies of fixed ratio is constant. A constant octave sweep rate is logarithmic sweep rate.

40. White Noise: White noise is noise of random nature having equal energy per unit frequency bandwidth over a specified total frequency band. White random vibration is synonymous with white noise. A white random vibration exists if the mean square acceleration density curve is of constant value over the specified total frequency band.

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